Lie Groups and Geometry, Sections 6-8

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Simon Donaldson

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Section 6: Spinors and exceptional Lie groups

The Spin representations
It can be shown that for a simply connected Lie group G of rank n there is a set of n weights $\omega_1, \dots, \omega_n$ in the FWC such that any weight in the FWC is a sum $\sum a_i \omega_i$ with integers $a_i \geq 0$. Let V_i be the irreducible representation corresponding to ω_i. It follows that any irreducible representation of G is contained in a tensor product of symmetric powers

$$s^{a_1}(V_1) \otimes \cdots \otimes s^{a_n}(V_n).$$

The V_i are called the *fundamental representations* of G.

For each ω_i there is a unique simple root α_i such that $\alpha_i . \omega_i \neq 0$.

So the nodes of the Dynkin diagram can be labelled by the fundamental representations.

For $G = SU(n+1)$ the fundamental representations are

$$V, \Lambda^2 V, \ldots, \Lambda^n V$$

where $V = \mathbf{C}^3$.
(We saw this before for $SU(3)$ since then $\Lambda^2 V = V^*$.)

Any representation is contained in a tensor power $V^{\otimes N}$.
There is an elaborate theory describing these representations.

For $G = Sp(n)$, let $V = \mathbf{C}^{2n}$ with standard symplectic form ω. Wedge product defines maps $L : \Lambda^i \to \Lambda^{i+2}$. The isomorphism $V = V^*$ gives maps $\Lambda : \Lambda^i \to \Lambda^{i-2}$.

For $i \leq n$ the “primitive” subspace P_i is the kernel of Λ. The fundamental representations are $P_1, \dots P_n$.

The orthogonal group $SO(m)$ has a double cover $\mathrm{Spin}(m)$ which is simply connected if $m > 2$.
The fundamental representations of $\mathrm{Spin}(2n+1)$ are

$$\Lambda^1, \ldots, \Lambda^{n-1}, S$$

and of $\mathrm{Spin}(2n)$ are

$$\Lambda^1, \ldots \Lambda^{n-2}, S^+, S^-$$

where S, S^+, S^- are the *spinor representations*.

Let

$$\widetilde{U}(n) = \{(g, a) \in U(n) \times S^1 : a^2 = \det g\}.$$

Projection to the first factor defines a double cover $\widetilde{U}(n) \to U(n)$.

Projection to the second factor defines a 1-dimensional representation L of $\widetilde{U}(n)$ such that $L^2 = \Lambda^n$.

Write

$$S = \left(\bigoplus \Lambda^i \right) \otimes L^{-1}.$$

This is a representation of $\widetilde{U}(n)$. We write $S = S^+ \oplus S^-$ according to i even or odd.

Proposition

$\widetilde{U}(n) \subset \mathrm{Spin}(2n)$ and the representations $S^{\pm}$ extend to irreducible representations of $\mathrm{Spin}(2n)$. Moreover there is an equivariant map

$$\Gamma : V \otimes S \to S$$

where $V = \mathbf{R}^{2n}$. Writing $\Gamma(\sigma \otimes v) = \gamma_v(\sigma)$ the γ_v map $S^{\pm}$ to $S^{\mp}$ and if $|v| = 1$ the map γ_v is an isometry with $\gamma_v^2 = -1$.

In fact the map Γ defines the $\mathrm{Spin}(2n)$ action.
For a vector space V (real or complex) with nondegenerate quadratic form Q the *Clifford algebra* $\mathrm{Cliff}(V)$ is the algebra generated by $1, V$ subject to the relation $v^2 = -Q(v)1$ for $v \in V$.
There is a canonical vector space isomorphism $\mathrm{Cliff}(V) = \Lambda^* V$.
Under this isomorphism, Clifford multiplication takes $\Lambda^2 \times \Lambda^2$ to $\Lambda^0 + \Lambda^2 + \Lambda^4$. The first and third components are symmetric and the second is skew symmetric. Thus the bracket $[a, b] = ab - ba$ maps $\Lambda^2 \times \Lambda^2$ to Λ^2 and defines a Lie algebra structure on Λ^2.
The basic fact is that, with suitable identifications, this is the same as the bracket on the Lie algebra of $SO(V, Q)$.

It follows that any representation of the Clifford algebra defines a representation of the Lie algebra of $SO(V,Q)$ which, by general theory, corresponds to a representation of the spin double cover (with a special treatment in the case $\dim V = 2$).

The algebra is a bit clearer in the complex case, so let now W be a complex vector space of dimension $2n$. We can assume that $V = U \oplus U^*$ with the quadratic form given by minus the dual pairing. Define $\Sigma = \Lambda^* U$. For $w \in W$ we define $\gamma_w : \Sigma \to \Sigma$ by:

- If $w = u \in U \subset W$ then $\gamma_u(\alpha) = u \wedge \alpha$;
- if $w = \eta \in U^* \subset W$ then $\gamma_\eta(\alpha) = i_\eta \alpha$.

Then $\gamma_u^2 = \gamma_\eta^2 = 0$ and

$$\gamma_u \gamma_\eta + \gamma_\eta \gamma_u = \eta(u) 1$$

So this gives a representation of the Clifford algebra on Σ.

We have $GL(U) \subset SO(W, Q)$. The Lie algebra action we have defined does not agree with the standard one on $\Lambda^* U$ but if we take $S = \Sigma \otimes L$ where L is a 1-dimensional representation of $\mathfrak{gl}(U)$ in which $\xi \in \mathfrak{gl}(U)$ acts as $\mathrm{Tr}(\xi)/2$, then the actions agree.

To get back to the real case, let V be a $2n$-dimensional real oriented Euclidean space and choose a compatible complex structure I on V. Set $W = V \otimes \mathbf{C}$. Then $W = V' \oplus V''$ where $I = i$ on V' and $-i$ on V'' and these are isotropic subspaces for the complex extension of the quadratic form, as above. Thus $S = \Lambda^* V' \otimes L$.

Our standard maximal torus in $U(n)$ gives a maximal torus in $SO(2n)$. The weight lattices of $\widetilde{U}(n)$ and $\mathrm{Spin}(2n)$ are the same, given by $\sum a_i\lambda_i$ where $a_i \in (1/2)\mathbf{Z}$ and are all equal modulo $\mathbf{Z}$. The 2^n weights of the representation S are

$$\pm\frac{1}{2}\lambda_1 \pm \frac{1}{2}\lambda_2 \cdots \pm \frac{1}{2}\lambda_n.$$

Those with an even/odd number of $+$ signs belong to $S^{\pm}$.

There is a complex antilinear map $* : \Lambda^p V' \to \Lambda^{n-p} V'$. This induces a $\mathrm{Spin}(2n)$-invariant antilinear map $\sigma : S \to S$.

- For n odd σ defines an isomorphism $S^- = \overline{S^+}$.
- For $n = 0$ mod 4 the representations $S^\pm$ are *real*.
- For $n = 2$ mod 4 the representations $S^\pm$ are *quaternionic*

Now let V be an oriented Euclidean space of dimension $2n-1$. The preceding discussion applies to $V \oplus \mathbf{R}e$ and gives spaces S_{2n}^+, S_{2n}^-. We define $S_{2n-1} = S_{2n}^+$. The map $\gamma_e : S_{2n}^+ \to S_{2n}^-$ is an isomorphism so we can use either.
On the other hand, if $V = V_{2n-2} \oplus \mathbf{R}e'$ we have defined spaces $S_{2n-2}^{\pm}$ and

$$S_{2n-1} = S_{2n-2}^+ \oplus S_{2n-2}^-.$$

The Clifford action of e' is by $+i$ on S_{2n-2}^+ and $-i$ on S_{2n-2}^-.

For $m = \pm 1 \mod 8$ the representation S_m is *real* and for $m = \pm 3 \mod 8$ it is quaternionic.

The co-adjoint orbit M corresponding to S^+_{2n} is the set $SO(2n)/U(n)$ of complex structures on $\mathbf{R}^{2n}$ compatible with metric and orientation. For S^- we reverse the orientation. In the complex description, M is one component of the set of n-dimensional isotropic subspaces in $(\mathbf{C}^{2n}, Q)$. For example, when $n = 2$ these correspond to the lines in a quadric surface in $\mathbf{CP}^3$: there two components given by the two rulings of a quadric surface.

In low dimensions the spin representations define the following isomorphisms:

- $\mathrm{Spin}(3) = SU(2) = Sp(1)$;
- $\mathrm{Spin}(4) = SU(2) \times SU(2) = Sp(1) \times Sp(1)$;
- $\mathrm{Spin}(5) = Sp(2)$;
- $\mathrm{Spin}(6) = SU(4)$.

Some exceptional Lie groups
Topics

1. G_2
2. Triality
3. F_4 and the Cayley plane
4. E_8.

In this subsection we write $\mathcal{S}_m$ etc. for the spin representation of $\mathrm{Spin}(m)$.
In dimension 7 the spin representation is a real vector space $\mathcal{S}_{7,\mathbf{R}}$ of dimension 8.

Proposition ExG1

$\mathrm{Spin}(7)$ *acts transitively on the unit sphere in* $\mathcal{S}_{7,\mathbf{R}}$.

Fix a decomposition $\mathbf{R}^7 = \mathbf{R}^6 + \mathbf{R}e'$ so $\mathcal{S}_7 = \mathcal{S}_6^+ \oplus \mathcal{S}_6^-$. Taking account of the real structure, $\mathcal{S}_{7,\mathbf{R}}$ is identified with $\mathcal{S}_6^+$, regarded as a real vector space.

We know that $\mathrm{Spin}(6) = SU(4)$. More precisely, the spin representation

$$\mathrm{Spin}(6) \to SU(\mathcal{S}_6^+)$$

is an isomorphism. It follows that $\mathrm{Spin}(6)$ acts transitively on the sphere in $\mathcal{S}_6^+$ and hence the Proposition.

Definition Fix a unit spinor $\sigma_0 \in \mathcal{S}_{7,\mathbf{R}}$. The Lie group G_2 is the stabiliser in $\mathrm{Spin}(7)$ of σ_0.

The dimension of $\mathrm{Spin}(7)$ is 21 so G_2 has dimension $21 - 7 = 14$.

The covering $\mathrm{Spin}(m) \to SO(m)$ has kernel $\{1, A\}$ say with $A^2 = 1$. One checks that A acts as -1 in the spin representation. In particular, $A \notin G_2 \subset \mathrm{Spin}(7)$ and hence G_2 can be regarded as a subgroup of $SO(7)$.

Proposition ExG2
G_2 acts transitively on the unit sphere S^6 in $\mathbf{R}^7$ and the stabiliser of a point is a copy of $SU(3) \subset SO(6) \subset SO(7)$.

Fix a unit vector e' in $\mathbf{R}^7$ as before and choose a complex structure on $\mathbf{R}^6$ so we have $\mathbf{R}^7 = \mathbf{R}e' \oplus \mathbf{C}^3$ and

$$\mathcal{S}_{7,\mathbf{R}} = (\Lambda^0 + \Lambda^2) \otimes \ (\Lambda^3)^{-1/2}.$$

Fix a basis element ν for $(\Lambda^3)^{-1/2}$ and let

$$\sigma_0 = 1 \otimes \nu \in \mathcal{S}_{7,\mathbf{R}}.$$

Use this to define G_2. We see that $SU(3) \subset G_2 \cap SO(6)$. The stabiliser in $SU(4)$ of a unit vector in $\mathbf{C}^4$ is $SU(3)$ so we see that $G_2 \cap SO(6) = SU(3)$.

We want to show that the derivative of the action of G_2 on S^6

$$D : \mathrm{Lie}(G_2) \to TS^6_{e'}$$

is surjective. By definition, the kernel of D is the Lie algebra of $G_2 \cap SO(6)$ which has dimension 8, as above. Thus the image has dimension $14 - 8 = 6$ and so is the whole tangent space.

We have $\mathrm{Lie}(G_2) = \mathfrak{su}(3) \oplus \mathbf{C}^3$ where the adjoint action restricts to the standard action of $SU(3)$ on $\mathbf{C}^3$. Here $\mathbf{C}^3$ is regarded as a real vector space. It follows that a maximal torus in $SU(3)$ is maximal in G_2. The 12 roots of G_2 are

- the 6 roots of SU(3), which have length $\sqrt{3}$;
- the 6 weights of the complex representation $\mathbf{C}^3 + \overline{\mathbf{C}}^3$, which have length 1.

Triality, I

In dimension 8 we have 8-dimensional real vector spaces $\mathcal{S}^{\pm}_{8,\mathbf{R}}$.

The spin representation gives a homomorphism $\mathrm{Spin}(8) \to SO(\mathcal{S}^{+}_{8,\mathbf{R}})$. The groups have the same dimension and since $\mathfrak{so}(8)$ is simple this must be a local isomorphism, which then lifts to an isomorphism $\mathrm{Spin}(8) \to \mathrm{Spin}(\mathcal{S}^{+}_{8,\mathbf{R}})$. It follows that there is an inner automorphism of $\mathrm{Spin}(8)$ which takes the fundamental representation on $\mathbf{R}^8$ to the $+$ spin representation. Similarly for the $-$ spin representation.

Later we will describe these explicitly.

F_4

Recall that a symmetric Lie algebra can be written $\mathfrak{g} \oplus \mathfrak{p}$ where $\mathfrak{g}$ is a subalgebra and the component of the bracket mapping $\mathfrak{p} \times \mathfrak{p}$ to $\mathfrak{p}$ is zero.

Suppose given a Lie algebra $\mathfrak{g}$ with invariant positive quadratic form and a representation on a Euclidean space $\mathfrak{p}$.

- The Lie algebra structure on $\mathfrak{g}$ is a map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.
- The action gives us a map $\mathfrak{g} \times \mathfrak{p} \to \mathfrak{p}$. Write this as $(\xi, p) \mapsto [\xi, p]$.
- Changing the sign, we have a map $\mathfrak{p} \times \mathfrak{g} \to \mathfrak{p}$, $[p, \xi] = -[\xi, p]$
- Using the Euclidean structures we have a map $\mathfrak{p} \times \mathfrak{p} \to \mathfrak{g}$, written $(p, p') \mapsto [p, p']$, defined by

$$\langle [p, p'], \eta \rangle = -\langle p', [p, \eta] \rangle.$$

The fact that the representation is Euclidean implies that this is skew-symmetric.

Set $X = \mathfrak{g} \oplus \mathfrak{p}$. The above data defines $[\ ,\ ] : X \times X \to X$ with the $\mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$ component set to zero. Conversely, any symmetric Lie algebra (with invariant definite form) arises this way.

When is $(X, [\ ,\ ])$ a Lie algebra?

The condition is that $\{x, y, z\} = 0$ for all $x, y, z \in X$ where

$$\{x, y, z\} = [[x, y], z] + [[y, z], x] + [[z, x], y].$$

- If $x, y, z \in \mathfrak{g}$ this holds since $\mathfrak{g}$ is a Lie algebra.
- If two of x, y, z are in $\mathfrak{g}$, say x and y, and $z \in \mathfrak{p}$ the condition is

$$[[x, y], z] = [x, [y, z]] - [y[x, z]],$$

which holds because $\mathfrak{p}$ is a representation of $\mathfrak{g}$.
- If one of x, y, z is in $\mathfrak{g}$, say x, and $y, z \in \mathfrak{p}$ the condition is

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]]$$

which holds because $[\ ,\ ] : \mathfrak{p} \otimes \mathfrak{p} \to \mathfrak{g}$ is a map of $\mathfrak{g}$ representations.

So the only potential problem comes when $x, y, z \in \mathfrak{p}$, in which case $\{x, y, z\}$ is also in $\mathfrak{p}$.
For "most" pairs $(\mathfrak{g}, \mathfrak{p})$ this will not be zero.

Proposition ExG3

Let $\mathfrak{g} = \mathfrak{so}(9)$ *and* $\mathfrak{p} = \mathcal{S}_{9,\mathbf{R}}$. *Then* $X, [\ ,\ ])$ *is a Lie algebra.*

As above, we need to check that $\{\sigma_1, \sigma_2, \sigma_3\} = 0$ for all $\sigma_i \in \mathcal{S}_{9,\mathbf{R}}$.

Consider $\mathfrak{so}(8) \subset \mathfrak{so}(9)$. We have, as usual,

$$\mathfrak{so}(9) = \mathfrak{so}(8) \oplus V$$

where $V = \mathbf{R}^8$ is the standard 8-dimensional representation.
This is a symmetric pair.
To streamline notation we now write $S^{\pm}$ for $\mathcal{S}^{\pm}_{8,\mathbf{R}}$. So
$\mathcal{S}_{9,\mathbf{R}} = S^{+} \oplus S^{-}$ and

$$X = \mathfrak{so}(8) \oplus V \oplus S^{+} \oplus S^{-}. \qquad (*****)$$

In the action of $\mathfrak{so}(9)$ on $S^{+} \oplus S^{-}$ the subalgebra $\mathfrak{so}(8)$ preserves $S^{\pm}$ while $V \subset \mathfrak{so}(9)$ interchanges them.

So the bracket on X maps $S^{+} \times S^{-}$ to V and $S^{\pm} \times S^{\pm}$ to $\mathfrak{so}(8)$.

If $\sigma_1, \sigma_2, \sigma_3 \in S^+$ then the calculation of $\{\sigma_1, \sigma_2, \sigma_3\}$ takes place within $\mathfrak{so}(8) \oplus S^+$.
Using triality $\mathfrak{so}(8) \oplus S^+$ is equivalent to $\mathfrak{so}(8) \oplus V$.

Since we know the latter is a Lie algebra we get $\{\sigma_1, \sigma_2, \sigma_3\} = 0$ in this case. Similarly if $\sigma_i \in S^-$.

We reduce to checking the case when $\sigma_1 \in S^+$ and $\sigma_2, \sigma_3 \in S^-$. Then $\{\sigma_1, \sigma_2, \sigma_3\} \in S^+$.

For fixed $\sigma_2, \sigma_3 \in S^-$, define maps $A, B : S^+ \to S^+$ by

$$A(\sigma_1) = [[\sigma_2, \sigma_3], \sigma_1],$$

$$B(\sigma_1) = [\sigma_2, [\sigma_3, \sigma_1]] - [\sigma_3, [\sigma_2, \sigma_1]].$$

We want to show that $A = B$.
By construction, our bracket on X satisfies

$$\langle x, [y, z] \rangle = -\langle [y, x], z \rangle$$

for the inner product on X.
Using this, we see that A, B are skew-symmetric maps, so we have $A, B \in \Lambda^2 S^+$.
Putting back the σ_2, σ_3 dependence, we now have maps $\alpha, \beta : \Lambda^2 S^- \to \Lambda^2 S^+$ with $\alpha(\sigma_2 \wedge \sigma_3) = A_{\sigma_2,\sigma_3}$ etc.

Clearly α, β are maps of $\mathfrak{so}(8)$ representations. By straightforward arguments $\Lambda^2 S^{\pm}$ are isomorphic irreducible representations of $\mathfrak{so}(8)$, in fact isomorphic to $\mathfrak{so}(8) = \Lambda^2 V$.

So α, β are equal up to a factor and to show $\alpha = \beta$ we just need to calculate one case. (Exercise)

Given Proposition ExG3, we have a compact simply connected Lie group F_4 with Lie algebra X. It contains $\mathrm{Spin}(8)$ and $\mathrm{Spin}(9)$ subgroups.

F_4 has dimension $28 + 3.8 = 52$. The maximal torus in $\mathrm{Spin}(8)$ remains maximal in F_4. The roots of F_4 are

- $\pm\lambda_i \pm \lambda_j$ $(i \neq j)$ 24 roots of length $\sqrt{2}$.
- $\pm\lambda_i$ 8 roots of length 1
- $\frac{1}{2}(\pm\lambda_1 \pm \lambda_2 \pm \lambda_3 \pm \lambda_4)$ 16 roots of length 1.

The pair $(F_4, \mathrm{Spin}(9)$ is symmetric so we have a compact Riemannian symmetric space $Z = F_4/\mathrm{Spin}(9)$ of dimension 16. It is the *Cayley plane.*

Proposition ExG4

$\mathrm{Spin}(9)$ *acts transitively on the unit sphere in* $\mathcal{S}_{9,\mathbf{R}}$*.*

Proof: Exercise.

Given a unit vector $e' \in \mathbf{R}^9$ we get a decomposition $\mathcal{S}_{9,\mathbf{R}} = S^+_{e'} \oplus S^-_{e'}$.

Proposition ExG5

For each unit spinor $\sigma \in \mathcal{S}_{9,\mathbf{R}}$ there is a unique unit vector $e' \in S^8 \subset \mathbf{R}^9$ such that $\sigma \in S^+_{e'}$.

Proposition ExG4 gives existence. For uniqueness consider a pair of linearly independent unit vectors e', e'' spanning a plane $\mathbf{R}^2$. Let e_1, e_2 be a orthonormal basis for this plane. One finds that

$$\mathcal{S}_{9,\mathbf{R}} = \mathcal{S}_{7,\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C},$$

where if $e'_\theta = \cos\theta e_1 + \sin\theta e_2$

$$S^+_{e'(\theta)} = \mathcal{S}_{7,\mathbf{R}} \otimes \mathbf{R}e^{i\theta}.$$

This establishes the Proposition.

Define a map $h : S^{15} \to S^8$ by $h(\sigma) = e'$ where $\sigma \in S^+_{e'}$.

This map h is a fibration with fibre S^7.

Recall that in a symmetric space with tangent space modelled on $\mathfrak{p}$ the sectional curvature in a pair of orthogonal vectors p_1, p_2 is

$$K(p_1, p_2) = \frac{1}{4}|[p_1, p_2]|^2.$$

In our case $\mathfrak{p} = \mathcal{S}_{9,\mathbf{R}}$.
For a unit vector $p_1 = \sigma \in S^+ = S^+_{e'}$, as above, the orthogonal complement in $\mathfrak{p}$ is $S^- \oplus N$ where N is the orthogonal complement of p_1 in S^+.
Calculation shows that (after suitable scaling) $K(p_1, p_2) = 1$ for $p_2 \in N$ and $= 1/4$ for $p_2 \in S^-$.

For $r < \pi$ the exponential map $\exp : S^- \to Z$ is an embedding on the r-ball. We get an induced Riemannian metric on the sphere S^{15}. As $r \to \pi$ the metric collapses the fibres of h and the metric limit is S^8.

The picture is the same as that for the complex and quaternionic projective planes with the Hopf maps $S^3 \to S^2$ and $S^7 \to S^4$.

Triality, II

We outline another proof of Proposition ExG3 which also sheds light on the symmetries involved. Let Γ be the permutation group on three elements.

Proposition ExG6

There is an action of Γ on X which preserves $[\ ,\ ]$ and which permutes transitively the three summands V, S^+, S^-.

Given this, to see that $\{x, y, z\} = 0$ for $x, y \in S^+$ and $z \in S^-$ it is equivalent to see it when $x, y \in V$ and $z \in S^+$ (say).

We know the latter by the definition of $[\ ,\ ]$.

Go back to $G_2 \subset \mathrm{Spin}(7) \subset \mathrm{Spin}(8)$.

We have a fixed unit vector $e \in \mathbf{R}^8$ and spinor $\sigma_+ \in S^+$. Let $\sigma_- = \gamma_e(\sigma_+) \in S^-$. Write S_0^+, S_0^- for the orthogonal complements of $\sigma_\pm \in S^\pm$.

Clifford multiplication $v \mapsto \gamma_v(\sigma_+)$ defines an isomorphism $R^7 \to S_0^+$ and similarly for S_0^-. Using these isomorphisms, Clifford multiplication $\mathbf{R}^7 \times S^+ \to S^-$ becomes a cross product

$$\times : \mathbf{R}^7 \times \mathbf{R}^7 \to \mathbf{R}^7.$$

One way to write this cross product explicitly is to choose a decomposition $\mathbf{R}^7 = \mathbf{R}e' \oplus \mathbf{C}^3$ as above. The symmetry group of $\mathbf{C}^3$ is $SU(3)$.

- For $v \in \mathbf{C}^3$, $e' \times v = Iv$.
- For $v, w \in \mathbf{C}^3$

$$v \times w = \omega(v, w)e' + v \times_{\mathbf{C}^3} w$$

where, ω is the metric 2-form and, in standard co-ordinates,

$$(v \times_{\mathbf{C}^3} w)_i = \sum \epsilon_{ijk} \overline{v}_j \overline{v}_k$$

.

The *Cayley algebra* (or *Octonion algebra*) $\mathbf{O}$ is the 8-dimensional non-associative algebra defined from this cross product in the same way as the quaternion algebra is defined from the usual cross product on $\mathbf{R}^3$.

When the symmetry group is restricted to G_2, each of $\mathbf{R}^8, S^+, S^-$ can be identified with $\mathbf{O}$. (More precisely, with an algebra isomorphic to $\mathbf{O}$.)

We also have a skew symmetric map $* : \mathbf{R}^7 \times \mathbf{R}^7 \to \mathfrak{g}_2$ defined using the action, as we have seen before.

We have $\mathfrak{so}(8) = \mathfrak{so}(7) \oplus \mathbf{R}^7_2$ and $\mathfrak{so}(7) = \mathfrak{g}_2 \oplus \mathbf{R}^7_1$ so

$$\mathfrak{so}(8) = \mathfrak{g}_2 \oplus \mathbf{R}^7_1 \oplus \mathbf{R}^7_2. \qquad (*****)$$

where $\mathbf{R}^7_i$ are copies of the standard representation and the equality is as representations of $\mathfrak{g}_2$.

Write (*****) as

$$\mathfrak{so}(8) = \mathfrak{g}_2 \oplus \mathbf{R}^7 \otimes \Pi \qquad (******)$$

where Π is a 2-dimensional Euclidean space with an orthonormal basis n_1, n_2 corresponding to the factors in (*****).

The component of the bracket in $\mathfrak{so}(8)$ mapping $\mathbf{R}^7 \otimes \Pi \times \mathbf{R}^7 \otimes \Pi$ to $\mathfrak{g}_2$ is the tensor product of the symmetric inner product on Π and the skew-symmetric $* : \mathbf{R}^7 \times \mathbf{R}^7 \to \mathfrak{g}_2$.

We know that the component of [,] from $\mathbf{R}^7_2 \times \mathbf{R}^7_2$ to $\mathbf{R}^7_2$ vanishes. Similarly for the component $\mathbf{R}^7_1 \times \mathbf{R}^7_1 \to \mathbf{R}^7_2$.
Let $\circ : \Pi \times \Pi \to \Pi$ be the symmetric bilinear map defined by

$$n_1 \circ n_1 = n_1 \ , \ n_2 \circ n_2 = -n_1 \ , \ n_1 \circ n_2 = -n_2.$$

Then some calculation shows that the component of [,] mapping $\Pi \otimes \mathbf{R}^7 \times \Pi \otimes \mathbf{R}^7$ to $\Pi \otimes \mathbf{R}^7$ is the tensor product of $\circ$ and $\times$.

Hence any linear map $A : \Pi \to \Pi$ which preserves the inner product and $\circ$ defines an automorphism of $\mathfrak{so}(8)$, equal to the identity on $\mathfrak{g}_2$.

Let $f(x, y)$ be a homogeneous polynomial on $\mathbf{R}^2$ of degree 3. Then the second derivatives of f are linear functions which are identified with points in $\mathbf{R}^2$ using the Euclidean structure. So f defines a symmetric bilinear map $\circ_f : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}^2$.
Let

$$f(x, y) = \frac{1}{6}\left(x^3 - 3xy^2)\right).$$

So $f_{xx} = x, f_{xy} = -y, f_{yy} = -x$. Then $\circ_f$ agrees with $\circ$ if we identify n_1, n_2 with the standard basis elements.

If $z = x + iy$ then

$$f(x, y) = \frac{1}{6}\mathrm{Re}(z^3),$$

which is clearly preserved by a Euclidean action of the group Γ.

Now write

$$X = \mathfrak{g}_2 \oplus \left(\Pi \otimes \mathbf{R}^7\right) \oplus \left(\mathbf{O} \otimes \mathbf{R}3\right),$$

and let e_i be the standard basis in $\mathbf{R}^3$. We have a skew-symmetric map

$$\mathbf{O} \otimes \mathbf{R}^3 \times \mathbf{O} \otimes \mathbf{R}^3 \to \mathbf{O} \otimes \mathbf{R}^3$$

defined by

$$((Z_1, Z_2, Z_3), (W_1, W_2, W_3)) \mapsto$$

$$(Z_2 W_3 - W_2 Z_3, Z_3 W_1 - W_3 Z_1, Z_1 W_2 - W_1 Z_2).$$

We also have a skew symmetric map

$$\mathbf{O} \otimes \mathbf{R}^3 \times \mathbf{O} \otimes \mathbf{R}^3 \to \mathbf{R}^7 \otimes \mathbf{R}^3$$

defined by

$$((Z_1, Z_2, Z_3), (W_1, W_2, W_3)) \mapsto \mathrm{Im}(Z_1 \overline{W}_1, Z_2 \overline{W}_2, Z_3 \overline{W}_3).$$

Compose with the Γ-equivariant projection map $\mathbf{R}^3 \to \Pi$ taking e_1 to n_2 to get

$$\mathbf{O} \otimes \mathbf{R}^3 \times \mathbf{O} \otimes \mathbf{R}^3 \to \mathbf{R}^7 \otimes \Pi.$$

Finally, we have our usual map

$$\mathbf{O} \otimes \mathbf{R}^3 \times \mathbf{O} \otimes \mathbf{R}^3 \to \mathfrak{g}_2,$$

defined using $*$ on $\mathbf{R}^7 = \mathrm{Im}\mathbf{O}$.
Putting these together, we get a bracket on X which is preserved by the action of the group Γ.
Now one has to check that this agrees with the bracket we defined before.

The group F_4 is the (connected) isometry group of the Riemannian manifold, Z, the Cayley plane. The subgroup $\mathrm{Spin}(8)$ fixes a triangle in Z with vertices p_1, p_2, p_3 say. The subgroup $\mathrm{Spin}(9)$ is the stabiliser of p_1. In our construction there are two other copies of $\mathrm{Spin}(9)$ visible in F_4. These are the stabilisers of p_2, p_3 so all three subgroups in F_4 are conjugate. The triality outer automorphisms of $\mathrm{Spin}(8)$ are induced by inner automorphisms of F_4.

There is an analogous situation for the complex projective plane $\mathbf{CP}^2 = SU(3)/U(2)$. Let T^2 be the maximal torus in $U(2)$. Then

$$\mathfrak{u}(2) = \mathfrak{t}_2 \oplus \mathbf{C},$$

while

$$\mathfrak{su}(3) = \mathfrak{u}(2) \oplus \mathbf{C}^2,$$

so

$$\mathfrak{su}(3) = \mathfrak{t}_2 \oplus \mathbf{C} \oplus \mathbf{C} \oplus \mathbf{C}.$$

The Weyl group of $SU(3)$ acts on $\mathfrak{t}_2$ and acts on the other three factors by permutation. We get three copies of $U(2)$ in $SU(3)$ which are the the stabilisers of the three vertices of a triangle.

For the quaternionic projective plane
$\mathbf{HP}^2 = Sp(3)/Sp(2) \times Sp(1)$ we have

$$\mathfrak{sp}(3) = \mathfrak{g} \oplus \mathbf{R}^4 \oplus \mathbf{R}^4 \oplus \mathbf{R}^4$$

where $G = Sp(1) \times Sp(1) \times Sp(1)$.
From these descriptions we see isometric embeddings $\mathbf{CP}^2 \subset \mathbf{HP}^2 \subset Z$ (and we could start with $\mathbf{RP}^2 \subset \mathbf{CP}^2$).

E_8

We can use a similar approach to build the Lie algebra of the exceptional Lie group E_8. Start with two copies of $\mathfrak{so}(8)$. We have

$$\mathfrak{so}(16) = \mathfrak{so}(8)_1 \oplus \mathfrak{so}(8)_2 \oplus V_1 \otimes V_2$$

where V_1, V_2 are the fundamental 8-dimensional representations. This is a symmetric decomposition (with associated symmetric space the Grassmanian of 8-planes in $\mathbf{R}^{16}$).

Now consider the real positive spin representation $\mathcal{S}^+_{16,\mathbf{R}}$ of $\mathfrak{so}(16)$. This has dimension 128. We can write it as

$$\mathcal{S}^+_{16,\mathbf{R}} = S_1^+ \otimes S_2^+ \oplus S_1^- \otimes S_2^-$$

where $S_i^{\pm}$ are the 8-dimensional real representations of $\mathfrak{so}(8)_i$.

Proceeding as before, we define a bracket on

$$X = \mathfrak{so}(8)_1 \oplus \mathfrak{so}(8)_2 \oplus V_1 \otimes V_2 \oplus S_1^+ \otimes S_2^+ \oplus S_1^- \oplus S_2^- .$$

Just as before, we can use triality to show that the Jacobi identity is satisfied, moving calculations into $\mathfrak{so}(16)$, which we understand.

Proposition ExG7

There is a compact connected Lie group E_8 of dimension 248 with Lie algebra X and a symmetric space $E_8/\mathrm{Spin}(16)$ of dimension 128.

The group E_8 has subgroups the other two exceptional Lie groups E_6, E_7.

Section 7: Unitary representations of $SL(2, \mathbf{R})$.

The study of infinite-dimensional unitary representations of non-compact Lie groups is a huge area.

Questions of *analysis* become important.

Two features.

(a) We cannot always decompose representations as direct sums, instead we need integrals. For example consider the group $(\mathbf{R}, +)$. The irreducible representations are 1-dimensional $\rho_\xi(x) = e^{i\xi x}$. The Hilbert space $L^2(\mathbf{R})$ is an infinite-dimensional representation and is decomposed as a direct integral of 1-dimensional representations via the Fourier transform

$$f(x) = (2\pi)^{-1/2} \int \hat{f}(\xi) e^{i\xi x} \; d\xi.$$

However the functions $e^{i\xi x}$ are not in L^2.

(b) We cannot pass so easily between Lie group representations and Lie algebra representations. For example **R** acts on itself by translation and hence on the functions on **R**. The derivative of the action is $D = \frac{d}{dx}$. The formula

$$\exp t(D) = 1 + tD + t^2 D^2/2 + \dots,$$

becomes the Taylor series formula

$$f(x+t) = f(x) + tf'(x) + \frac{t^2}{2} f''(x) + \dots$$

which holds (for small t) only if f is real analytic.

Similarly, we cannot always complexify actions. For example, taking the complex valued functions on **R**, a complexification of the **R** action would involve solving the Cauchy-Riemann equation

$$\frac{\partial f}{\partial \tau} = i \frac{\partial f}{\partial t},$$

with given initial condition at $\tau = 0$. This is not a well-posed PDE problem.

In our short discussion we largely avoid questions of analysis. Our aim is to:

- Define some unitary representations of the group $SL(2, \mathbf{R})$;
- Make it at least plausible that this a list of all irreducible representations (Bargmann classification, 1947).
- Discuss some interesting geometry related to these representations.

Definitions

Let G be a Lie group. A *unitary representation* of G is a representation $\rho : G \to U(V)$, where V is a complex Hilbert space, such that for each $v \in V$ the map $g \mapsto \rho(g)v$ is continuous.

A unitary representation is *irreducible* if there are no non-trivial closed G-invariant subspaces.

We only consider $G = SL(2, \mathbf{R})$. Recall that this is isomorphic to $SU(1,1)$ and $\mathrm{Spin}(2,1)$ (the double cover of $SO(2,1)$).

The maximal compact subgroup is $K = S^1$ and G/K is the hyperbolic plane $\mathcal{H}$.

The isomorphism $SL(2, \mathbf{R}) = SU(1,1)$ is reflected in the upper half-plane and disc models of hyperbolic geometry.

Lie algebra discussion.

Recall that the Lie algebra $\mathfrak{su}(1,1)$ of $SU(1,1)$ is the set of matrices

$$\begin{pmatrix} ia & \alpha \\ \overline{\alpha} & -ia \end{pmatrix}$$

with $a \in \mathbf{R}, \alpha \in \mathbf{C}$. The complexification is $\mathfrak{sl}(2,\mathbf{C})$ in which we have our standard basis:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

$$[H,X] = 2X, [H,Y] = -2Y, [X,Y] = H. \qquad (****)$$

We want to analyse the possibilities for the following data:

1. a collection of finite-dimensional complex vector spaces V_k for $k \in \mathbf{Z}$. Let $\underline{V}$ be the space (possibly infinite dimensional) of all *finite sums*
$$\underline{V} = \bigoplus V_k.$$
2. An action of $\mathfrak{sl}(2, \mathbf{C})$ on $\underline{V}$ such that H acts with weight k on V_k.
3. $\underline{V}$ is irreducible, in the sense that there is no $\mathfrak{sl}(2, \mathbf{C})$ invariant proper subspace.
4. Hermitian structures on the V_k such that the subalgebra $\mathfrak{su}(1, 1)$ maps to skew-adjoint operators on $\underline{V}$ (with the inner products between the V_k set to zero).

Remark:There is an important theorem which states that any irreducible unitary representation of $SU(1, 1)$ contains a dense subspace of the form $\underline{V}$, as above.

Arguing on familiar lines one sees that conditions (1),(2),(3) require

- $\dim V_k \leq 1$ (to see this use the *Casimir operator*, see below).
- The k for which $V_k \neq 0$ are either all even or all odd.

Still using (1),(2),(3) one finds that there are five possibilities.

I V is finite-dimensional.

II $V_k \neq 0$ for all even k.

III $V_k \neq 0$ for all odd k.

IV There is an $l \geq 0$ such that $V_k \neq 0$ for $k \geq l$ and $k = l \bmod 2$.

V There is an $l \leq 0$ such that $V_k \neq 0$ for $k \leq l$ and $k = l \bmod 2$.

We have already analysed case (I) and the spaces do not satisfy condition (4) so we ignore it.

Now set $\xi = X + Y, \eta = i(X - Y)$ so iH, ξ, η is a basis for $\mathfrak{su}(1,1)$.

Consider first case (IV). Suppose $l = 0$ and choose a basis element $v_0 \in V_0$. Since $[X, Y]v_0 = 0$ we have $YXv_0 = 0$ and $\xi X v_0 = X^2 v_0$ so for any choice of norms $\langle \xi X v_0, v_0 \rangle = 0$ but

$$\langle X v_0, \xi v_0 \rangle = |X v_0|^2.$$

So the action of ξ cannot be skew-adjoint and we conclude that $l = 0$ is impossible.

Still in case (IV), suppose $l = 1$ and choose a unit-norm basis element $v_1 \in V_1$. The $X^j v_1$ for $j \geq 0$ form a basis for $\underline{V}$.

We have $-YXv_1 = [X, Y]v_1 = 2v_1$ so

$$\langle \xi X v_1, v_1 \rangle = \langle (X^2 + YX) v_1, v_1 \rangle = -2|v_1|^2,$$

and the skew adjoint condition gives

$$|Xv_1|^2 = 2|v_1|^2 = 2.$$

Continuing in the same way one finds that for each j the norm of $X^j v_1$ is fixed by condition (4) and conversely the norms so determined satisfy (4).
Similarly, in case (IV) for each $l \geq 1$ and in case (V) for each $l \leq -1$, there is a unique irreducible Hermitian Lie algebra representation $\underline{V}$. The representations for $l, -l$ are complex conjugate.

Now consider case (II) and choose a unit-norm vector $e \in V_0$. Define $\lambda \in \mathbf{C}$ by $YXe = -\lambda e$. Then one sees by induction that for $k \geq 1$ $YX^k e = -\lambda_k X^{k-1} e$ with

$$\lambda_k = \lambda + 2(1 + \cdots + (k-1)) = \lambda + k(k-1).$$

Suppose we have a compatible norm with $|X^k e|^2 = h_k$. Then

$$\langle \xi X^k e, X^{k-1} e\rangle = -\langle X^k e, \xi X^{k-1} e\rangle = -\langle X^k e m X^k e\rangle = -h_k.$$

On the other hand

$$\langle \xi X^k e, X^{k-1} e\rangle = \langle YX^k e, X^{k-1} e\rangle = -\lambda_k |X^{k-1} e|^2 = -\lambda_k h_{k-1}.$$

So we need λ_k to be real and positive for all k i..e $\lambda > 0$.

Conversely, for any $\lambda > 0$ there is a unique irreducible Hermitian Lie algebra representation $\underline{V}$ of type (II).

There is a similar discussion for case (III), with the exception that for one value of the parameter we get a reducible representation, the sum of those of type (IV),(V) with $l = \pm 1$.

Construction of representations

The induced representation construction.

In general let G be a Lie group and $H \subset G$ a subgroup. Let σ be a representation of H on a vector space W and $M = G/H$. Regarding $G \to M$ as a principle H-bundle we get an associated vector bundle $E_\sigma \to M$ with fibre W. This is a G equivariant bundle so G acts on the space of sections of E. Depending on the context, we can consider sections of various kinds (continuous, smooth, holomorphic, ...). We call these induced representations.

For example if G is compact and $T \subset G$ is a maximal torus then we have seen that all irreducible representations of G are obtained as induced from 1-dimensional representation of T, restricting to *holomorphic* sections over the complex manifold M.

The representations of $SL(2,\mathbf{R})$ are all induced representations with the subgroups $SO(2) \subset SL(2,\mathbf{R})$ and P, the matrices

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix},$$

with $a \neq 0$.
In the first case the construction is similar to that in the case of compact groups. It is convenient to work with $S^1 \subset SU(1,1)$ so G/S^1 is the disc model of the hyperbolic plane $\mathcal{H}$. The line bundles associated to the representations of S^1 are the fractional powers $K^{m/2}$ of the canonical bundle $K = T^*\mathcal{H}$.
We have the $SU(1,1)$-invariant Poincaré metric on $\mathcal{H}$:

$$ds^2 = \frac{1}{(1-|z|^2)^2}(dx^2 + dy^2).$$

Definition

For $m \geq 2$, D_m is the space of L^2 holomorphic sections of $K^{m/2}$ where the L^2 norm is the standard one defined by the Poincaré metric.

Explicitly, D_m can be regarded as the expressions $f(z)dz^{m/2}$ on the disc with f holomorphic and

$$\int |f|^2(1-|z|^2)^{m-2} < \infty.$$

The element $e^{i\theta}$ in $S^1 \subset SU(1,1)$ acts on the disc by $z \mapsto e^{2i\theta}z$. So it acts on $z^a dz^{m/2}$ as multiplication by $e^{ik\theta}$ with $k = m + 2a$. This gives the *discrete series* representations corresponding to case (II) with $l = m \geq 2$.

For case (III) we take complex conjugates, expressions $f(\overline{z})d\overline{z}^{m/2}$, to get representations $\overline{D}_m$.

The obvious definition of D_m does not work if we put $m = 1$. We will return to that case below.

Now go back to $G = SL(2, \mathbf{R})$ and the subgroup P. Then G/P is the real projective line $\mathbf{RP}^1 = \mathbf{R} \cup \{\infty\}$.

For any $\zeta \in \mathbf{C}$ we have a representation of $P \to \mathbf{C}^*$ defined by

$$\rho_\zeta \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = |a|^{2\zeta}.$$

This defines a complex line bundle $\Lambda_\zeta \to \mathbf{RP}^1$.

If $\zeta = 1/2 + is$, with $s \in \mathbf{R}$, there is an invariant L^2 norm on the sections of Λ_ζ.

To see this recall that over any manifold M there is a bundle $\mathcal{D}$ of *densities*. This is a bundle with fibre $\mathbf{R}$ and structure group $\mathbf{R}^+$ acting by multiplication. In terms of a covering of M by co-ordinate charts, the transition functions for $\mathcal{D}$ are given by the absolute values of the Jacobians of the co-ordinate change maps. A section of $\mathcal{D}$ defines a measure on M.

For any $\eta \in \mathbf{C}$ we can form the complex power $\mathcal{D}^\eta$.

Recall from algebraic geometry that if E is a 2-dimensional vector space with a fixed volume element in $\Lambda^2 E$ there is a canonical isomorphism $T^*\mathbf{P}(E) = \mathcal{O}(-2)$. In our situation this means that $\Lambda_{1/2} = \mathcal{D}^{1/2}$ and $\Lambda_\zeta = \mathcal{D}^\zeta$. Explicitly we can write a section of Λ_ζ in terms of an affine coordinate x as

$$f(x)\ |dx|^\zeta.$$

We have $\Lambda_{\overline{\zeta}} = \overline{\Lambda_\zeta}$. If s_1, s_2 are sections of Λ_ζ then $s_1\overline{s_2}$ is a section of $\Lambda_{(\zeta+\overline{\zeta})}$ which is $\mathcal{D}$ if $\zeta = 1/2 + is/2$. Thus

$$\int_{\mathbf{RP}^1} s_1\overline{s_2}$$

is well-defined.

Definition

The principle series representation P_s of $SL(2,\mathbf{R})$ is that on L^2 sections of $\Lambda_\zeta \to \mathbf{RP}^1$, with $\zeta = 1/2 + is/2$.

From another point of view we could regard P_s as $L^2(\mathbf{R})$ but with action

$$(A^{-1}f)(x) = |cx+d|^{-(1+is)} f\left(\frac{ax+b}{cx+d}\right)$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

We have other representations $P \to \mathbf{C}^*$ given by

$$\rho_\zeta^- \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \mathrm{sgn}\,(a)\,|a|^{2\zeta}.$$

The resulting line bundle over $\mathbf{RP}^1$ is $\Lambda_\zeta \otimes_{\mathbf{R}} \Lambda^-$ where Λ^- is the Möbius band line bundle with structure group ± 1.

Proceeding in just the same way we get another principle series P_s^-

Example There is an obvious unitary representation of $SL(2,\mathbf{R})$ on $L^2(\mathbf{R}^2)$. We claim that

$$L^2(\mathbf{R}^2) = \int_{-\infty}^{\infty} P_s\, ds \oplus \int_{-\infty}^{\infty} P_s^-\, ds \qquad (**)$$

where the two summands correspond to even/odd functions on $\mathbf{R}^2$.

Take standard polar co-ordinates (r,θ) on $\mathbf{R}^2$ and set $r = e^{t/2}$. Then

$$\|f\|_{L^2(\mathbf{R}^2)} = \frac{1}{2}\int_{-\infty}^{\infty} e^t |f(t,\theta)|^2 dt d\theta.$$

The map $f \mapsto r^{1/2} f$ defines an equivalence

$$L^2(\mathbf{R}^2) = L^2(\mathbf{R}\times S^1).$$

Take the Fourier transform in the **R** variable. This gives a representation, reverting to the r variable:

$$f(r,\theta) = r^{-1/2} \int_{s=-\infty}^{\infty} \tilde{f}(s,\theta) r^{-is/2} = \int_{-\infty}^{\infty} \tilde{f}(s,\theta) r^{-(1/2+is/2)}.$$

The circle here is the double cover of $\mathbf{RP}^1$. We can represent $\tilde{f}(s, \ )$ as a sum $\tilde{f}_s^+ + \tilde{f}_s^-$ where $\tilde{f}^\pm$ are sections of the trivial bundle and $\Lambda^- \otimes \mathbf{C}$ respectively over $\mathbf{RP}^1$.

The above construction is manifestly $SO(2)$ invariant. When the symmetry group is restricted to $SO(2)$ we have an invariant volume form on $\mathbf{RP}^1$ so the bundles Λ_ζ are trivialised. So we can choose to interpret $\tilde{f}_s^\pm$ as sections of $\Lambda_\zeta, \Lambda_\zeta \otimes \Lambda^-$ respectively. The point is that with this interpretation the construction is $SL(2,\mathbf{R})$-invariant and this gives (**).

Recall that the invariant λ of a Lie algebra representation of type (II) is defined by $YX = -\lambda e$ where e is a basis element in V_0.

Proposition

For the representation P_s the invariant is $\lambda = \zeta - \zeta^2$ where $\zeta = \frac{1}{2}(1 + is)$.

Consider $\mathbf{RP}^1$ as the unit circle in $\mathbf{C}$. Then one finds that $\xi, \eta \in \mathfrak{su}(1,1)$ correspond to the vector fields

$$2\cos\theta\partial_\theta \quad , \quad 2\sin\theta\partial_\theta$$

respectively on the circle.

The diffeomorphisms of the circle act on the sections of the bundle Λ_ζ so there is a Lie derivative. For a vector field $v = a(\theta)\partial\theta$ and section $s = f(\theta)|d\theta|^\zeta$ the formula is

$$L_v(s) = \left(a\frac{df}{d\theta} + \sigma f\frac{da}{d\theta}\right)|d\theta|^\zeta.$$

To simplify notation write $f(\theta)|d\theta|^{\zeta}$ as $f(\theta)$. Then

$$\xi(f) = 2\left(\cos\theta\frac{df}{d\theta} - \zeta f\sin\theta\right)$$

and

$$\eta(f) = 2\left(\sin\theta\frac{df}{d\theta} + \zeta f\cos\theta\right).$$

Now $X = (\xi + i\eta)/2$ and $Y = (\xi - i\theta)/2$ so

$$X(f) = e^{-i\theta}(\frac{df}{d\theta} - i\zeta f) \ \ Y(f) = e^{i\theta}(\frac{df}{d\theta} + i\zeta f).$$

In this notation the element e is the constant function $f = 1$. We find

$$YX(1) = \zeta^2 - \zeta$$

so $\lambda = \zeta - \zeta^2$.

Write $\zeta - \zeta^2 = \frac{1}{4} - (\zeta - 1/2)^2$. If $\zeta = \frac{1}{2}(1 + is)$ with s real we get all values $\lambda \geq 1/4$.
If $\zeta = 1/2 + \sigma/2$ with σ real and $0 < \sigma < 1$ we also have λ real. The corresponding representations C_σ form the *complimentary series*. Now it is less obvious how to define an $SL(2, \mathbf{R})$ invariant norm.

In algebro-geometric language the diagonal in $\mathbf{P}^1 \times \mathbf{P}^1$ is a divisor in the linear system $\mathcal{O}(1, 1)$. There is an SL_2 invariant section of $\mathcal{O}(-2, -2)$ with a double pole on the diagonal. Identifying $\mathcal{O}(1)$ with $K^{-1/2}$ and using affine coordinates this is

$$\Gamma = (x_1 - x_2)^{-2} dx_1 dx_2.$$

Then for any η we have an $SL(2, \mathbf{R})$-invariant object

$$|\Gamma|^\eta = |x_1 - x_2|^{-2\eta} |dx_1|^\eta |dx_2|^\eta.$$

Now for s_1, s_2 sections of $\Lambda_{1/2+\sigma/2}$, for σ as above, we define a pairing $\langle s_1, s_2\rangle$ (linear in the first factor and antilinear in the second) as follows. If $s_1 = f|dx|^{1/2+\sigma/2}, s_2 = g|dx|^{1/2+\sigma/2}$

$$\langle s_1, s_2\rangle = \int\int f(x_1)\overline{g}(x_2)|dx_1|^{1/2+\sigma/2}|dx_2|^{1/2+\sigma/2}|\Gamma|^{(1/2-\sigma/2)}.$$

That is:

$$\int\int f(x_1)\overline{g}(x_2)\frac{1}{|x_1 - x_2|^{\sigma-1)}}dx_1 dx_2.$$

The integral is defined initially for smooth sections. The fact that this defines a norm can be proved using Fourier Transforms. The representation C_σ is defined to be the Hilbert space completion.

A similar construction gives an operator defining an isomorphism $P^{\pm}_{-s} = \overline{P^{\pm}_s}$.

Go back to D_1.
There is a well-defined restriction map from sections of $K^{1/2}$ (defined initially over a slightly larger disc) to sections of $\Lambda^-_{1/2}$.
For any curve $\gamma(t)$ the map is defined locally by

$$f(z)dz^{1/2} \mapsto f(\gamma(t))\sqrt{\gamma'(t)}|dt|^{1/2}.$$

Going around the circle the square root changes sign so we map to $\Lambda^-_{1/2}$.
The representation D_1 is defined to be the completion of the half-forms holomorphic on a slightly larger disc, using the norm of the boundary value in P_0^-.
This also defines an invariant subspace $D_1 \subset P_0^-$ and in fact

$$P_0^- = D_1 \oplus \overline{D}_1.$$

Theorem (Bargmann)
The irreducible unitary representations of $SL(2, \mathbf{R})$ are:

- *The principal series P_s for $s \geq 0$;*
- *The odd principal series P_s^- for $s > 0$;*
- *The discrete series $D_n, \overline{D}_n$ $(n \geq 2)$;*
- *The "mock" discrete series $D_1, \overline{D}_1$;*
- *The complimentary series C_σ $(0 < \sigma < 1)$*

and these are all distinct.

Remarks

- There are some connections between these representations and the co-adjoint orbits of $SL(2, \mathbf{R})$, but not as straightforward as for compact groups.
- The constructions easily extend to certain other groups. For example $SU(n, 1)$ acts on the unit ball in $\mathbf{C}^n$ and we can consider L^2 holomorphic forms.

Section 8: Eigenfunctions and the Selberg Trace formula

In the upper half space model, the Laplace operator on $\mathcal{H}$ is

$$\Delta\phi = -y^2(\phi_{xx} + \phi_{yy}).$$

If $\phi = y^\zeta$ then $\Delta\phi = \lambda\phi$ with $\lambda = \zeta - \zeta^2$. Applying the map $z \mapsto -z^{-1}$ we get another eigenfunction

$$\left(\frac{y}{x^2 + y^2}\right)^\zeta.$$

For a function f on $\mathbf{R}$ we define a function ϕ_f on $\mathcal{H}$

$$\phi_f(x, y) = \int_{-\infty}^{\infty} \left(\frac{y}{(x - x')^2 + y^2}\right)^\zeta f(x')dx',$$

which is an eigenfunction, with the same λ.

Proposition *The map taking sections of* Λ_ζ *to functions on* $\mathcal{H}$ *defined by* $f(x)|dx|^\zeta \mapsto \phi_f$ *is* $SL(2, \mathbf{R})$ *equivariant.*

Thus we can regard the principle series P_s and complimentary series C_ζ as spaces of eigenfunctions on $\mathcal{H}$.

For another point of view, use the model of $\mathcal{H}$ as one sheet $x_0 > 0$ of the hyperboloid $q(x_1, x_2, x_3) = 1$ where $q = x_0^2 - x_1^2 - x_2^2$. Suppose that F is a solution of the wave equation

$$\left(\partial_0^2 - \partial_1^2 - \partial_2^2\right) F = 0$$

on the positive cone which is homogeneous of degree ζ i.e.

$$F(\rho \underline{x}) = \rho^\zeta F(\underline{x})$$

for $\rho > 0$, then the restriction of F to $\mathcal{H}$ is an eigenfunction of the Laplacian with eigenvalue $\zeta(\zeta - 1) = \zeta^2 - \zeta$.

Let n be a null vector for the quadratic form q and $h_n(x) = (x, n)$ for the symmetric form $(\, , \,)$ corresponding to q. Then for any f the function $F(x) = f(h_n(x))$ is a solution of the wave equation. This is just the fact that in $1+1$ dimensions any function $f(x-t)$ satisfies the wave equation.

If n is in the component $x_0 > 0$ of the null cone then h_n is positive on the positive cone and the function h_n^ζ yields an eigenfunction on $\mathcal{H}$.

This makes it clear that there is an equivariant map from sections of Λ_ζ to eigenfunctions.

For a third point of view we recall the notion of the Casimir operator.

Suppose $\mathfrak{g}$ is a Lie algebra with a nondegenerate symmetric form. In a basis e_α of $\mathfrak{g}$ write $g_{\alpha\beta} = \langle e_\alpha, e_\beta \rangle$. Let $\left(g^{\alpha\beta}\right)$ be the inverse matrix. Given a representation of $\mathfrak{g}$ on a vector space V the Casimir operator $C : V \to V$ is

$$\sum_{\alpha\beta} g^{\alpha\beta} e_\alpha \circ e_\beta,$$

where $\circ$ is the composition in $\mathrm{End}\ V$.

This has the property that it commutes with action of $\mathfrak{g}$ so on an irreducible representation it must be constant.

In the case of $\mathfrak{sl}(2, \mathbf{C})$ the Casimir operator is

$$-(1/2)(H^2 + XY + YX).$$

Restricted to $\mathfrak{su}(1, 1)$ we can express this as

$$-\frac{1}{2}(\xi^2 + \eta^2 - h^2)$$

where $h = iH$.

Going back to our Lie algebra discussion we see that the scalar invariant λ for type II is given by the Casimir operator. Thus on sections of Λ_ζ it acts as $\zeta - \zeta^2$.

One sees also that, acting on functions on $\mathcal{H}$, the Casimir operator is the Laplace operator Δ.

Sections of $\Lambda_\zeta \to \mathbf{RP}^1$ can be regarded as functions on $G = SL(2, \mathbf{R})$ which transform appropriately under the right action of the subgroup P. We also have the right action of the circle K. Integrating over the K-orbits gives a map $C^\infty(G) \to C^\infty(G/K)$.
Putting these together we get a G-equivariant map

$$I : \Gamma(\Lambda_\zeta) \to C^\infty(G/K) = C^\infty(\mathcal{H}).$$

This must be compatible with the Casimir operators so we see that for any section s of Λ_ζ the function $I(s)$ is an eigenfunction of the Laplacian on $\mathcal{H}$ with eigenvalue $\zeta - \zeta^2$.

Some Harmonic analysis on $\mathcal{H}$.
Recall

An L^1 function $K(x)$ on $\mathbf{R}^n$ defines a convolution operator T_K. Under Fourier transforms this goes over to a multiplication operator by $\hat{K}(\xi)$. If K is a function of $r = |x|$ then $\hat{K}$ is a function of $\rho = |\xi|$. If f satisfies $\Delta_{\mathbf{R}^n} f = \rho^2 f$ then

$$T_K(f) = \hat{K}(\rho) f.$$

We want the analogous theory for functions on $\mathcal{H}$.

Let k be a function on $\mathbf{R}^+$ with suitable decay at infinity. For $x, y \in \mathcal{H}$, let $\underline{k}(x, y) = k(d(x, y))$ where $d(x, y)$ is the distance in $\mathcal{H}$.

Define T_k on functions on $\mathcal{H}$ by

$$T_k(f)(x) = \int_{\mathcal{H}} \underline{k}(x, y) f(y) dy.$$

Proposition

For each λ there is a $P(\lambda)$ such that if f satisfies $\Delta f = \lambda f$ then $T_k(f) = P(\lambda) f$.

The map taking the function k to the function P is the analogue of the Fourier transform on rotationally invariant functions.

To prove the Proposition pick a base point x_0 in $\mathcal{H}$ with isotropy group $S^1 \subset SU(1,1)$. By ODE theory, for any λ there is a unique smooth S^1-invariant solution F_λ to $\Delta F_\lambda = \lambda F$ with $F_\lambda(x_0) = 1$. Define

$$P(\lambda) = T_k(F_\lambda)(x_0).$$

Now let f be any function with $\Delta f = \lambda f$. Let $\underline{f}$ be obtained by averaging f over rotations by S^1. It is clear that

$$T_k(f)(x_0) = T_k(\underline{f})(x_0).$$

On the other hand $\underline{f}$ must be a multiple of F_λ so

$$\underline{f} = f(x_0) F_\lambda.$$

Then $T_k(f)(x_0) = f(x_0)\ T_k(F_\lambda)(x_0) = P(\lambda) f(x_0)$.
Clearly the same applies, with the same $P(\lambda)$, for any $x_0 \in \mathcal{H}$.

We want a formula for $k \mapsto P$.
In the half-plane model, use the function $f = y^{\zeta}$ which satisfies $\Delta f = \lambda f$ for $\lambda = \zeta - \zeta^2$. Take the base point $x_0 = i$. We have

$$P(\lambda) = \int_{\mathcal{H}} y^{\zeta-2} k(d(x+iy), i) \quad dxdy.$$

In general, for points z, w in the half-plane define

$$D(z,w) = \frac{|z-w|^2}{\mathrm{Im}\, z \ \mathrm{Im}\, w}$$

Then $1 + D(z,w) = \cosh d(z,w)$. Write $\kappa(D) = k(\cosh^{-1}(1+D))$. Then

$$P(\lambda) = \int_{\mathcal{H}} \kappa(y^{-1}(x^2 + (y-1)^2)) y^{\zeta-2} dxdy.$$

Write $x = y^{1/2}u, y = e^{2t}$ and $\zeta = 1/2 + is/2$. We get:

$$P(\lambda) = 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \kappa(u^2 + 4\sinh^2(t))e^{ist} du dt.$$

Define an operator (the Abel transform) on functions on $\mathbf{R}$ by

$$A(f)(v) = \int_{-\infty}^{\infty} f(v + u^2)\, du.$$

Then, if $\lambda = s^2 + 1/4$,

$$P(\lambda) = 2\int_{-\infty}^{\infty} A(\kappa)(4\sinh^2 t)e^{ist} dt.$$

So the procedure to go from κ to P is the composite of

- The Abel transform $\kappa \mapsto A(\kappa)$;
- change of variable $v = 4\sinh^2 t$;
- take the Fourier transform at s, where $\lambda = s^2/4 + 1/4$.

Let ∂ denote the operation of differentiation, on functions on $\mathbf{R}$. The operator A commutes with ∂ and satisfies:

$$A^2\partial = -\pi \mathrm{id}.$$

up to a factor.
To see this, for a function f we have

$$A^2(f)(v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v + u_1^2 + u_2^2)\, du_1 du_2.$$

Take polar co-ordinates $u_1 = r\cos\theta, u_2 = r\sin\theta$. Then

$$A^2(f)(v) = \int_0^{\infty}\int_0^{2\pi} f(v + r^2) r dr d\theta$$

which is

$$\pi \int_0^{\infty} f(v + \rho) d\rho.$$

So for a function f vanishing at infinity

$$\left(A^2\partial\right)(f)\,(v) = -\pi f(v).$$

Thus $A^{-1} = -\pi^{-1} A\partial = -\pi^{-1}\partial A$. All the steps above can be inverted and we have a procedure to go from the function P to the function κ.

- Set $h(s) = P(s^2/4 + 1/4)$.
- Take the inverse Fourier transform:

$$g(t) = (2\pi)^{-1} \int_{-\infty}^{\infty} h(s) e^{-ist} \ ds.$$

- Change variables to $v = 4\sinh^2 2t$ and, for $v \geq 0$ set $Q(v) = g(t)$. Write $Q' = \frac{dQ}{dv}$.
- Then

$$\kappa(D) = \int_0^{\infty} Q'(D + u^2)\ du.$$

We will be particularly interested in $\kappa(0) = k(0)$.

$$\kappa(0) = \int_0^\infty Q'(u^2)du = \int_0^\infty Q'(v)\frac{dv}{2\sqrt{v}}$$

We have

$$\frac{dQ}{dv} = \frac{dg}{dt}\frac{dt}{dv}$$

so

$$\kappa(0) = \int_0^\infty \frac{dg}{dt}\frac{dt}{2\sqrt{v}} = \int_0^\infty \frac{dg}{dt}\frac{dt}{4\sinh t}.$$

Now

$$\frac{dg}{dt} = (2\pi)^{-1}\int_{-\infty}^\infty (-is)h(s)e^{-ist}\ ds.$$

So, bearing in mind that h is an even function of s,

$$\kappa(0) = \int_0^\infty \int_0^\infty s\ h(s)\frac{\sin(st)}{\sinh t}\ dsdt.$$

By contour integration one can show that

$$\int_0^\infty \frac{\sin st}{\sinh t} dt = \tanh(2s)$$

So (*up to a factor !*):

$$\kappa(0) = \int_0^\infty s \ \tanh(2s) \ h(s) \ ds.$$

Taking the theory further there is a decomposition of representations:

$$L^2(\mathcal{H}) = \int_0^\infty s \tanh(2s) P_s. \quad (***)$$

To give some suggestion towards this, recall that if M is a compact space and K is a continous function on $M \times M$ the integral operator

$$T_K(f)(x) = \int_M K(x,y) f(y)\, dy,$$

is a trace-class operator and the trace is

$$\mathrm{Tr}(T_K) = \int_M K(x,x)\, dx.$$

If we apply this formally to T_k on $\mathcal{H}$ we would have

$$\mathrm{Tr}(T_k) = k(0)\mathrm{Vol}(\mathcal{H}).$$

Of course the volume is infinite so this does not make literal sense.
But having in mind that T_k acts as $h(s)$ on P_s the other way to write the "trace" given (***) is

$$\mathrm{Tr}(T_k) = \int_0^\infty s \tanh(2s) h(s)\, ds\, \mathrm{dim} P_s.$$

Short digression

Under very general conditions on a group G there is a *Plancherel measure* μ on the set $\hat{G}$ of isomorphism classes of unitary representations such that

$$L^2(G) = \int_{\hat{G}} V_\rho \otimes V_\rho^* \, d\mu(\rho).$$

(Theorem of Naimark).

For $G = SL(2, \mathbf{R})$ the measure is supported on the series P_s, P_s^- and the discrete series $D_n, n \in \mathbf{Z}$. The formula is

$$L^2(G) = \bigoplus_n D_n \otimes D_n^* \oplus \int_0^\infty P_s \, s \, tanh(2s) \, ds \oplus \int_0^\infty P_s^- \; s \; \coth(2s) ds.$$

The only representations that have a K-fixed vector are the P_s and for each s that space is 1-dimensional. Considering the right action of K we see that the Placherel formula implies that

$$L^2(\mathcal{H}) = \int_0^\infty P_s \, s \tanh(2s) ds.$$

Conversely if we know this, and a corresponding statement in the "odd" case, we can recover the Plancherel formula by considering the operator ∂ on $\mathcal{H}$ taking sections of $K^{m/2}$ to sections of $K^{1+m/2}$.

The Selberg Trace formula
Background

First, let G be a compact Lie group and Γ a subgroup of G. Let $M = \Gamma \backslash G$. Then G acts on M and hence on $L^2(M)$. Let V_α be an irreducible representation of G.
What is the multiplicity m_α of V_α in $L^2(M)$?
We know that

$$L^2(G) = \bigoplus_\alpha V_\alpha \otimes V_\alpha^*.$$

So m_α is the dimension of the Γ-invariant subspace in V_α^*.

Now let G be any Lie group with bi-invariant measure and $\Gamma \subset G$ a discrete subgroup such that $M = \Gamma \backslash G$ is compact. Then $L^2(M)$ is a representation of G.
Then it can be shown that the irreducible unitary representations of G occur discretely in $L^2(M)$.
In particular take $G = SL(2, \mathbf{R})$. For simplicity, we assume that Γ maps injectively to $PSL(2, \mathbf{R}) = SL(2, \mathbf{R}) / \pm 1$. Suppose that Γ acts freely on $\mathcal{H}$. So $\Gamma \backslash \mathcal{H}$ is a compact Riemann surface

$$\Sigma = \Gamma \backslash G / K.$$

The manifold M is a circle bundle over Σ, corresponding to a square root $K_{\Sigma}^{1/2}$.

Similar to the compact case, but with a more involved proof, one has

Proposition

1. The multiplicity of P_s in $L^2(M)$ is the dimension of the space of eigenfunctions of Δ on Σ with eigenvalue $s^2/4 + 1/4$.
2. The multiplicity of D_n in $L^2(M)$ is the dimension of the space of holomorphic sections of $K_\Sigma^{n/2}$ on Σ.

With similar statements for the P_s^- and the complementary series.

Here we are concerned with case (1) above.

The same “spectrum” appears from Riemannian geometry and representation theory.

Write Λ for this Laplacian eigenvalue spectrum of Σ, counted with multiplicity in the usual way.

Let $P(\lambda)$ be a (suitable) function on $[0, \infty)$, for example $P(\lambda) = e^{-\lambda\tau}$. Then we can form an operator $P(\Delta)$ on Σ and

$$\mathrm{Tr}\, P(\Delta) = \sum_{\lambda \in \Lambda} P(\lambda).$$

The Selberg Trace formula expresses this trace in terms of other geometric data.

Definitions

A geodesic loop on Σ *is a geodesic* $\gamma : [0, L] \to \Sigma$ *with* $\gamma(0) = \gamma(L)$*.*

A primitive closed geodesic is the image of a geodesic embedding $S^1 \to \Sigma$*.*

Write $\mathcal{L}$ for the "length spectrum", the lengths of primitive closed geodesics, counted with multiplicity.

As before, set $h(s) = P(1/4 + s^2/4)$ and let $g(t)$ be the Fourier transform.

The Selberg Trace formula (for suitable functions P) is

$$\mathrm{Tr} P(\Delta) = (4\pi)^{-1} \mathrm{Area}(\Sigma) \int_0^\infty h(s) s \tanh(2s)\, ds + \sum_{L \in \mathcal{L}} \Pi(L),$$

where

$$\Pi(L) = L \sum_{m=1}^{\infty} \frac{g(mL/2)}{\sinh mL/2}.$$

For example take $P(\lambda) = e^{-\tau\lambda}$ for $\tau > 0$.

This defines the heat kernel on Σ. Consider the asymptotics as $\tau \to 0$. The first term on the right hand side of the formula has an asymptotic expansion $a_0\tau^{-1} + a_1 + a_2\tau + \dots$. This is what could be computed from local differential geometry. The sum in the second term involves terms like $\exp(-L^2/\tau)$ which vanish to infinite order as $\tau \to 0$.

To establish the trace formula, let k be the function corresponding to P, as discussed above.
For $x, y \in \mathcal{H}$ write $\underline{k}(x, y) = k(d(x, y))$. Then

$$K(x, y) = \sum_{\gamma \in \Gamma} \underline{k}(x, \gamma y) \qquad (*)$$

is preserved by Γ acting on x and y and so descends to a function K_Σ on $\Sigma \times \Sigma$ defining an operator T_Σ.
Let $\pi : \mathcal{H} \to \Sigma$ be the covering map. From the definition we have

$$T_k(\pi^*(f)) = \pi^*(T_\Sigma f).$$

Using what we know on $\mathcal{H}$, it follows that $T_\Sigma = P(\Delta)$. So

$$\mathrm{Tr} P(\Delta) = \int_\Sigma K_\Sigma(x, x)\, dx. \qquad (**)$$

Let $\widetilde{\Sigma}$ be the set of pairs $(x, [\alpha])$ where $x \in \Sigma$ and $[\alpha] \in \pi_1(\Sigma, x)$. For each such pair there is a unique geodesic loop α based at x in the given homotopy class. So we have a length function $\widetilde{L} : \widetilde{\Sigma} \to \mathbf{R}$.

From the definition, there is a covering map $p : \widetilde{\Sigma} \to \Sigma$, so $\widetilde{\Sigma}$ is a Riemann surface with hyperbolic metric. Looking at (*) we see that

$$K_\Sigma(x, x) = \sum_{\tilde{x} \in \pi^{-1}(x)} k(\widetilde{L}(\tilde{x})),$$

so

$$\int_\Sigma K_\Sigma(x, x)\, dx = \int_{\widetilde{\Sigma}} k(\widetilde{L}(\tilde{x}))\, d\tilde{x}. \qquad (***)$$

Let Ω be the set of conjugacy classes in $\pi_1(\Sigma)$. It can be identified with the free homotopy classes of maps $S^1 \to \Sigma$.

The space $\widetilde{\Sigma}$ is not connected, it has connected components $\widetilde{\Sigma}_a$ corresponding to classes a in Ω.

If $\alpha \in \pi_1(\Sigma)$ is a representative for a class $a \in \Omega$ then $\widetilde{\Sigma}_a = Z \backslash \mathcal{H}$ where $Z \subset \pi_1$ is the *centraliser* of α. (The centralisers of different representatives are conjugate so this is independent of the choice of α.)

Putting this together:

$$\mathrm{Tr}\, P(\Delta) = \sum_{a\in\Omega} I_a$$

where

$$I_a = \int_{\widetilde{\Sigma}_a} I(L(\tilde{x}))\, d\tilde{x}.$$

For the trivial class $a = 0$ we have $\widetilde{\Sigma}_0 = \Sigma$ and

$$I_0 = \int_\Sigma k(0)$$

which is the first term in the trace formula, by our calculation of $k(0)$.

An element of $PSL(2, \mathbf{R})$ which has no fixed points in $\mathcal{H}$ is conjugate to

$$\begin{pmatrix} \mu^{1/2} & 0 \\ 0 & \mu^{-1/2} \end{pmatrix}$$

for some $\mu > 1$. The centraliser is isomorphic to $\mathbf{R}$. It follows that if $\alpha \in \pi_1(\Sigma) = \Gamma$ is not the identity then the centraliser is isomorphic to $\mathbf{Z}$. If α is primitive then the centraliser is generated by α.

Suppose a is a primitive class. Then, from the above,

$$\widetilde{\Sigma}_a \cong \mathcal{H}/Z$$

where Z is the infinite cyclic group generated by $z \mapsto \mu z$ (in the upper half space model), for some μ. A fundamental domain is

$$\{z : 1 \leq \mathrm{Im}\, z \leq \mu\}.$$

From this one sees that *each primitive conjugacy class contains a unique primitive closed geodesic representative*. The parameter μ above is e^L where L is the length of the geodesic. Using the function κ as before we get, for a primitive class,

$$I_\alpha = \int_{-\infty}^{\infty}\int_1^{\mu} \kappa\left[S^2\frac{x^2+y^2}{y^2}\right] \; y^{-2}\, dxdy,$$

where $\mu = e^L$ and $S = \mu^{1/2} - \mu^{-1/2} = 2\sinh L/2$.
Change variables by $x = uy/S$ to get

$$I_\alpha = \frac{1}{2\sinh L/2}\int_{-\infty}^{\infty}\int_1^{\mu} \kappa[u^2 + 4\sinh^2(L/2)] \; du\, \frac{dy}{y}.$$

Hence

$$I_\alpha = \frac{L}{2\sinh L/2} g(L/2).$$

This gives the contribution from primitive classes in the trace formula (the term $m = 1$ in the sum defining $\Pi(L)$. A small variant of the calculation deals with the other classes (i.e. m times a primitive class).