Lie Groups and Geometry, Sections 1-5

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Section1. Review of some standard material

A Lie group is a smooth manifold endowed with a group structure such that multiplication and inversion are smooth maps.

A complex Lie group is a complex manifold endowed with a group structure such that multiplication and inversion are holomorphic maps.

If G is a Lie group the tangent space $\mathfrak{g}$ at the identity has the structure of a Lie algebra, with a skew symmetric bilinear map

$$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

satisfying the Jacobi identity

$[\xi_1, [\xi_2, \xi_3]] + [\xi_2, [\xi_3, \xi_1]] + [\xi_3, [\xi_1, \xi_2]] = 0.$

Some equivalent definitions of the bracket.

1. For any representation ρ of G on a vector space V we get a map $d\rho : \mathfrak{g} \to \mathrm{End}\, V$. Apply this to the *adjoint representation* of G on $\mathfrak{g}$, induced by the action of G on itself by conjugation: $d\rho(\xi)(\eta) = [\xi, \eta]$.
2. Identify $\mathfrak{g}$ with the *left-invariant vector fields* on G. The bracket on $\mathfrak{g}$ is induced by the Lie bracket on vector fields.
3. Choose any local coordinate system to identify a neighbourhood of 1_G in G with a neighbourhood of 0 in $\mathfrak{g}$. Then

$$\xi_1 \xi_1 = \xi_1 + \xi_2 + A(\xi_1, \ xi_2) + \text{higher order terms},$$

   and $[\xi_1, \xi_2]$ is the skew-symmetric part of A.

The bracket on $\mathrm{End}\ V$ is $[M_1, M_2] = M_1 M_2 - M_2 M_1$. The Jacobi identity can be read as the statement that $d\rho$ in (1) is a Lie algebra homomorphism.

For any ξ in $\mathfrak{g}$ there is a unique 1-parameter subgroup $t \mapsto \exp(t\xi)$ with derivative ξ at $t = 0$. The exponential map $\exp : \mathfrak{g} \to G$ gives a diffeomorphism from a neighbourhood of0 to a neighbourhood of 1_G.

If G is a compact Lie group then any representation of G, real or complex, admits an invariant Euclidean/Hermitian structure. In particular this is true for the adjoint representation. An invariant Euclidean structure on $\mathfrak{g}$ defines a bi-invariant Riemannian metric on G, preserved by left and right translation. The 1-parameter subgroups are then the geodesics through 1_G. Hence, or otherwise, one sees that the exponential map of a compact Lie group is surjective.

Example: the compact Lie group $SU(2)$. This consists of complex matrices

$$M = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix},$$

with $\det M = |\alpha|^2 + |\beta|^2 = 1$. As a manifold it can be identified with the sphere S^3. The 1-parameter subgroups are great circles, intersecting in antipodal points ± 1.

Let G be any Lie group and ρ a representation on a vector space V. The formula

$$B_\rho(\xi_1, \xi_2) = \mathrm{Tr}\ \rho(\xi_1)\rho(\xi_2),$$

defines an invariant symmetric form on $\mathfrak{g}$. In the case of the adjoint representation this is called the Killing form. If the representation is orthogonal, the form is negative and if also the representation is faithful it is negative definite.
Relevant parts of the above extends to complex Lie groups in a straightforward way. A representation of a complex Lie group is a holomorphic map from G to $GL(V)$ (Note that the only compact complex Lie groups are tori.)

Section 2: Compact real forms

A Lie algebra is called simple if it has no proper ideals (and is not one-dimensional).

Theorem 1

Let $\mathfrak{g}$ be a simple complex Lie algebra. There is a complex Lie group G with Lie algebra $\mathfrak{g}$ and a compact real subgroup K such that $\mathfrak{g}$ is the complexification of $\mathfrak{k}$: i.e. $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$.

(Remark: A complex Lie group which is the complexification of a compact Lie group is called *reductive*.)

Examples

- $G = SL(n, \mathbf{C}), K = SU(n)$
- $G = SO(n, \mathbf{C}), K = SO(n)$, $n \neq 2, 4$.
- $G = Sp(n, \mathbf{C}), K = Sp(n)$.

In the third item, $Sp(n, \mathbf{C}) \subset SL(2n, \mathbf{C})$ is the subset preserving a non-degenerate skew symmetric form. The compact form $Sp(n)$ is the set of *quaternionic* $n \times n$ matrices M with $MM^* = 1$. The two are connected bhy regarding $\mathbf{H}^n$ as $\mathbf{C}^{2n}$ with skew form $\langle x, Jy \rangle$.

We outline a (non-standard) geometric proof of Theorem 1 , which fits it into a more general framework of "Geometric Invariant Theory and the Kempf-Ness Theorem".

Let V be a complex vector space. A Lie algebra structure on V is a tensor in $\Lambda^2 V^* \otimes V$. We regard this as a representation of the group $SL(V)$. More generally, consider any representation ρ of $SL(V)$ on a complex vector space W. We can assume that there is a Hermitian metric on W preserved by the action of $SU(V) \subset SL(V)$. Let $w_0 \in W$ and consider its $SL(V)$-orbit $\mathcal{O} \subset W$. *Suppose* that there is a point $w \in \mathcal{O}$ which minimises the norm, among all points in $\mathcal{O}$. Let G' be the stabiliser of w in $SL(V)$ and $K' = G' \cap SU(V)$.

Lemma *The Lie algebra of G' is the complexification of the Lie algebra of K'.*

The Lie algebra homomorphism $d\rho : \mathfrak{sl}(V) \to \mathfrak{sl}(W)$ takes $\mathfrak{su}(V)$ to $\mathfrak{su}(W)$. This implies that $\rho(\xi^*) = \rho(\xi)^*$ where the adjoint $*$ has the usual meaning with respect to the metrics on V, W.

The hypothesis that w minimises the norm implies that $\mathrm{Re}\langle d\rho(\xi)(w), w\rangle = 0$ for all $\xi \in \mathfrak{sl}(V)$. Take $\xi = [\eta, \eta^*]$ for some $\eta \in \mathfrak{sl}(V)$ and set $M = d\rho(\eta)$. Then $d\rho([\eta, \eta^*]) = [M, M^*]$ and

$$\langle [M, M^*]w, w\rangle = |M^* w|^2 - |Mw|^2.$$

By definition, η lies in $\mathrm{Lie}(G')$ if and only if $d\rho(\eta) = 0$. So we see that $*$ preserves $\mathrm{Lie}(G')$ and, since $(*)^2 = 1$, $\mathrm{Lie}(G')$ is the sum of the ± 1 eigenspaces of $*$ which are $\mathrm{Lie} K'$ and $i\, \mathrm{Lie} K'$.

Apply this to our situation where w_0 is the bracket defining a simplex complex Lie algebra. Suppose we can find a minimising point as above. This defines an equivalent Lie algebra structure so without loss of generality $w = w_0$. We want to see that $\mathfrak{g} = \mathrm{Lie}(G')$. By definition, $\mathrm{Lie}(G')$ is the Lie algebra of *derivations*, i.e linear maps $\alpha : \mathfrak{g} \to \mathfrak{g}$ such that

$$\alpha([\xi_1, \xi_2] = [\alpha(\xi_1), \xi_2] + [\xi_1, \alpha(\xi_2)].$$

The adjoint representation defines a homomorphism $\xi \mapsto \mathrm{ad}_\xi$ from $\mathfrak{g}$ to $\mathrm{Lie}(G')$ and this is injective since $\mathfrak{g}$ is simple. The identity

$$[\alpha, \mathrm{ad}_\xi] = \mathrm{ad}_{\alpha(\xi)} \qquad (*)$$

shows that the image is an ideal in $\mathrm{Lie}(G')$.

Let $\mathfrak{g}^\perp$ be the orthogonal complement of $\mathfrak{g}$ in $\mathrm{Lie}(G')$ with respect to the Hermitian inner product. We claim that is also an ideal in $\mathrm{Lie}(G')$. For if $\alpha \in \mathrm{Lie}(G'), \beta \in \mathfrak{g}^\perp, \gamma \in \mathfrak{g}$ we have

$$\langle[\alpha,\beta],\gamma\rangle = \mathrm{Tr}([\alpha,\beta]^*\gamma) = -\mathrm{Tr}([\alpha^*,\beta^*]\gamma) = -\mathrm{Tr}(\alpha^*\beta^*\gamma - \beta^*\alpha^*\gamma.$$

Using the property of trace this is

$$-\mathrm{Tr}(\gamma\alpha^*\beta^* - \alpha^*\gamma\beta^*) = -\langle[\gamma,\alpha^*],\beta\rangle.$$

But we know that $\alpha^* \in \mathrm{Lie}(G')$ and $\mathfrak{g} \subset \mathrm{Lie}(G')$ is an ideal, so $[\gamma,\alpha^*] \in \mathfrak{g}$ and is orthogonal to β.
Now we have a direct sum of two ideals $\mathrm{Lie}(G') = \mathfrak{g} \oplus \mathfrak{g}^\perp$ so $[\mathfrak{g},\mathfrak{g}^\perp] = 0$. The identity (*) shows that for $\beta \in \mathfrak{g}^\perp$ $\mathrm{ad}_{\beta(\xi)} = 0$ for all $\xi \in \mathfrak{g}$, hence $\beta = 0$ and $\mathfrak{g} = \mathrm{Lie}(G')$.

Theorem 1 follows from a general result.

Theorem 2 *Let W be a representation of $SL(V)$ and w_0 a non-zero point in W. If there is no norm-minimising point in $\mathcal{O}$ then the stabiliser of w_0 in $SL(V)$ preserves a proper subspace of V.*

In our case, the stabiliser of w_0 contains a copy of $\mathfrak{g}$ and if this preserves a subspace of $\mathfrak{g}$ that subspace is an ideal. So Theorem 2 implies Theorem 1.

Our proof of Theorem 2 involves considerations of the homogeneous space $\mathcal{H} = SL(V)/SU(V) = SL(n, \mathbf{C})/SU(n)$ which can be identified with the space of positive definite self-adjoint $n \times n$ matrices with determinant 1. The identification is given by mapping $g \in SL(n, \mathbf{C})$ to $H = g^* g$. This space has a Riemannian metric preserved by the action of $SL(n, \mathbf{C})$:

$$\|\delta H\|_H^2 = \mathrm{Tr}(\delta H\, H^{-1})^2.$$

This metric has *(weakly) negative sectional curvature*. (Later we will see that this fits into a more general story concerning Riemannian symmetric spaces). The geodesics through the base point 1 have the form $\exp(ht)$ where h is self-adjoint and trace-free. Clearly the map

$$\exp : T\mathcal{H}_1 \to \mathcal{H}$$

is a diffeomorphism. A standard comparison theorem in Riemannian geometry states that this map is *distance-increasing*. In the case when $n = 2$ the space $\mathcal{H}$ can be identified with hyperbolic 3-space.

More generally we can consider any complete, simply connected, Riemannian manifold X of negative sectional curvature and the distance-increasing exponential map. We define an equivalence relation on geodesics in X by $\gamma_1 \sim \gamma_2$ if $d_X(\gamma_1(t), \gamma_2(t))$ is bounded as $t \to \infty$. The set of equivalence classes is the *sphere at infinity* $S_\infty(X)$. The group $\mathrm{Iso}(X)$ acts on $S_\infty(X)$.

In the case $X = \mathcal{H}$ the sphere at infinity can be identified with the unit sphere in the vector space of trace-free self adjoint matrices. For such a matrix M, let

$$\lambda_1 > \lambda_2 > \cdots > \lambda_r$$

be the eigenvalues with eigenspaces $P_1, \ldots, P_n$. Let

$$Q_1 = P_1 \ , \ Q_2 = P_1 \oplus P_2 \ , \ \ldots$$

Then $0 \subset Q_1 \subset Q_2 \cdots \subset \mathbf{C}^n$ is a *flag* in $\mathbf{C}^n$. We can recover the matrix M from the flag and the "weights" λ_i. So $S_\infty(\mathcal{H})$ can be identified with the set of weighted flags. The action of $SL(n, \mathbf{C}) \subset \mathrm{Iso}(\mathcal{H})$ on S_∞ is the obvious action on the weighted flags.

A function F on a manifold X as above is called convex if it is convex on geodesics in the usual sense.

Theorem 3 *For X as above, suppose F is a convex function preserved by a subgroup $\Gamma \subset \mathrm{Iso}(X)$. If F does not have a minimum in X there is a point in $S_\infty(X)$ fixed by Γ.*

We show that Theorem 3 implies Theorem 2 (hence Theorem 1). The function $\tilde{F}(g) = |g(w_0)|^2$ on $SL(n, \mathbf{C})$ descends to a function $F : \mathcal{H} \to \mathbf{R}$. It is preserved by the action of $\Gamma = \mathrm{Stab}(w_0) \subset SL(n, \mathbf{C})$ on $\mathcal{H}$. We claim that this function is convex on geodesics.

It suffices to consider a geodesic $\exp(tM)$ through the base point 1. Then $A = (d\rho)(M)$ is a self-adjoint endomorphism of W with eigenvalues μ_j say. Then

$$|\rho(\exp(tM))(w_0) = \sum \chi_j^2 e^{2\mu_j t}$$

where χ_j is the norm of the component of w_0 in the μ_j eigenspace of A. This function is clearly convex. (In fact $\log F$ is convex.)

Now Theorem 2 immediately follows from Theorem 3, since if Γ preseves a flag it certainly preserves some non-trivial subspace.

One proof of Theorem 3 goes via considering the negative gradient flow of the function F: i.e the integral curves of $\dot{x} = -\mathrm{grad} F(x)$. Let $x_1(t), x_2(t)$ be two solutions. The convexity of F implies that $d_X(x_1(t), x_2(t))$ is *decreasing*. One deduces easily that the flow exists for all positive time and that the geodesics from $x(0)$ to $x(t)$ have a limit in $S_\infty(X)$ which is independent of the choice of flow line. All the constructions are invariant under Γ so this limit is fixed by Γ.

In the Geometric Invariant Theory picture the point we have found in S_∞ corresponds to the "optimal destabilising 1-parameter subgroup".

There is a variant of the discussion above for real Lie algebras.

Theorem 4

Let $\mathfrak{g}$ be a simple real Lie algebra. There is a Lie group G with Lie algebra $\mathfrak{g}$, a faithful representation $G \to SL(V)$ and a Euclidean metric on V such that the image of $\mathfrak{g}$ in $\mathrm{End}\, V$ is preserved by transposition. The intersection $K = G \cap SO(V)$ is a compact subgroup of G and any compact subgroup of G is conjugate to a subgroup of K. In particular, the subgroup K is a maximal compact subgroup of G and is unique up to conjugation.

The proof of Theorem 4 is the same as for Theorem 1, with minor changes.

Reference *Lie algebra theory without algebra* arxiv 0702016

A Lie algebra $\mathfrak{g}$ with an involution $\sigma : \mathfrak{g} \to \mathfrak{g}$ is called *symmetric*. The ± 1 eigenspaces of σ give a vector space decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k} \ , \ [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p} \ , \ [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

We get another Lie algebra structure $\mathfrak{g}'$ on the same vector space by changing the sign of the component $\mathfrak{p} \times \mathfrak{p} \to \mathfrak{k}$.
In the situation of Theorem 4, we get such a structure with $\sigma(\xi) = -\xi^T$. Then $\mathfrak{k}$ is the Lie algebra of the compact group K.
The subspaces $\mathfrak{k}, \mathfrak{p}$ are orthogonal with respect to the Killing form, which is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$.
The Lie algebra $\mathfrak{g}'$ is the Lie algebra of a compact group G'.

The homogeneous spaces $G/K, G'/K$ are *dual symmetric spaces*. The involution σ of the Lie algebras induces one of the Lie groups. Up to coverings, the quotients $G/K, G'/K$ can be realised as submanifolds of the groups: the fixed points of the maps $\tau(g) = \sigma(g)^{-1}$.

The sectional curvature of a bi-invariant Riemannian metric on a Lie group is given by $\frac{1}{4}|[\xi_1, \xi_2]|^2$. This formula works equally well for a "metric" of indefinite signature. The submanifolds $G/K \subset G, G'/K \subset G'$ are totally geodesic submanifolds (because τ is an isometry and τ^2 is the identity. It follows that G/K has negative sectional curvature and G'/K has positive sectional curvature.

Examples

- $G = SO(p,q), K = S(O(p) \times O(q)), G' = SO(p+q)$. The homogeneous spaces are Grassmann manifolds. (In particular when $q = 1$ we get spherical and hyperbolic geometries.)
- $G = SL(n, \mathbf{R}), K = SO(n), G' = SU(n)$. The symmetric space G/K is the set of Euclidean structures on $\mathbf{R}^n$ with fixed determinant. The dual space G'/K is the set of "special Lagrangian" subspaces in $\mathbf{C}^n$.
- $G = Sp(n, \mathbf{R}), K = U(n), G' = Sp(n)$. G/K is the set of complex structures on $\mathbf{R}^{2n}$ compatible with a fixed symplectic form. G'/K is the set of n-dimensional complex Lagrangian subspaces of $\mathbf{H}^n = \mathbf{C}^{2n}$.
- G is the complexification K^c of a compact group K. Then $G' = K \times K$ and the symmetric spaces are K^c/K and K. When $K = SU(n)$ we get the space $\mathcal{H}$ used in the proof of Theorem 1.

The symmetric spaces G/K of negative type fit into the framework of Theorem 3 and we get a result like Theorem 2 for more general group actions.

Section 3. Some structure theory for compact Lie groups

A compact connected abelian Lie group is a torus.

Theorem 5 *Let G be a compact Lie group. Any element of G lies in some torus subgroup $T \subset G$. Up to conjugation there is a unique maximal torus.*

The dimension of a maximal torus is called the *rank* of the group.

Example
$G = SU(n)$: a maximal torus is the set T of diagonal matrices in $SU(n)$. The rank is $(n-1)$.

Observe: "Most" (i.e. outside a set of measure 0) elements of $SU(n)$ lie in a unique maximal torus. The exceptions are those with multiple eigenvalues.

Lemma *Let* $\xi, \xi' \in \mathfrak{g}$*. There is a* $g \in G$ *such that* $[g(\xi), \xi'] = 0$*.* (Here we are writing $g(\xi)$ for the adjoint action of G on $\mathfrak{g}$.)
Proof. Fix an invariant Euclidean form on $\mathfrak{g}$. Since G is compact we can maximise the function $g \mapsto \langle g(\xi), \xi' \rangle$ over $g \in G$. Without loss of generality, the maximum is achieved at $g = 1$. The maximality implies that $\langle [\eta, \xi], \xi' \rangle = 0$ for all $\eta \in \mathfrak{g}$. But

$$\langle [\eta, \xi], \xi' \rangle = \langle \eta, [\xi, \xi'] \rangle$$

and so $[\xi, \xi'] = 0$.

If T is a torus then a generic element ξ in $\mathrm{Lie}\, T$ generates T in the sense that T is the closure of the 1 parameter subgroup $\exp(t\xi)$.

It is a general fact that any closed subgroup of a Lie group is a Lie subgroup.

Since the exponential map for G is surjective, any element of G lies in some torus subgroup.

Suppose $T, T' \subset G$ are two maximal tori and choose generators $\xi, \xi' \in \mathfrak{g}$. Apply the Lemma: after replacing T by a conjugate subgroup we can suppose that $[\xi, \xi'] = 0$. The fact that ξ, ξ' are generators implies that T, T' commute and maximality implies that $T = T'$.

The Lie algebra $\mathfrak{t}$ of a maximal torus is called a *Cartan subalgebra* of $\mathfrak{g}$. It is a real vector space with a Euclidean structure and an integer lattice $\Lambda \subset \mathfrak{t}$ defined by $T = \mathfrak{t}/\Lambda$.

The main point of the theory is to understand the Lie group G via geometry in this Euclidean space $\mathfrak{t}$.

The irreducible complex representations of T are 1-dimensional, corresponding to a dual lattice $\Lambda^* \subset \mathfrak{t}^*$. For $\alpha \in \Lambda^*$ the representation takes $\exp(\xi)$ to multiplication by $e^{i\alpha(\xi)}$. (The definition of Λ^* is that $\alpha(\xi) \in 2\pi\mathbf{Z}$ for $\xi \in \Lambda$.)

α is called the *weight* of the representation.

Let $\mathfrak{g}_{\mathbf{C}}$ be the complexification $\mathfrak{g} \otimes \mathbf{C}$. The restriction of the adjoint action of G makes it a complex representation of T. The trivial part is just $\mathfrak{t}_{\mathbf{C}} = \mathfrak{t} \otimes \mathbf{C}$. The non-trivial weights appearing are called the *roots*. Thus

$$\mathfrak{g}_{\mathbf{C}} = \mathfrak{t}_{\mathbf{C}} \oplus \bigoplus_{\alpha} R_\alpha$$

where α runs over the roots. The fact that the representation is the complexification of a real representation means that the roots come in pairs $\pm\alpha$.

$\mathfrak{g}_{\mathbf{C}}$ is a complex Lie algebra. By construction $[\mathfrak{t}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}}] = 0$ and for $\xi \in \mathfrak{t}_{\mathbf{C}}, r_\alpha \in R_\alpha$

$$[\xi, r_\alpha] = i\alpha(\xi)\eta,$$

for the complex linear extension of α to $\mathfrak{t}_{\mathbf{C}}$.

We have a FUNDAMENTAL CALCULATION of the theory: If $r_{\alpha'} \in R_{\alpha'}$ then for $\xi \in \mathfrak{t}_{\mathbf{C}}$:

$$[\xi, [r_\alpha, r_{\alpha'}] = [r_{\alpha'}, [r_\alpha, \xi]] - [r_\alpha, [r_{\alpha'}, \xi]]$$

which is

$$[r_{\alpha'}, -i\alpha(\xi)r_\alpha] - [r_\alpha, -i\alpha'(\xi)r_{\alpha'}] = i(\alpha + \alpha')(\xi)[r_\alpha r_{\alpha'}].$$

We conclude that

- if $\alpha \neq -\alpha'$ then either $[R_\alpha, R_{\alpha'}] = 0$ or $\alpha + \alpha'$ is also a root and $[R_\alpha, R_{\alpha'}] \subset R_{\alpha+\alpha'}$;
- $[R_\alpha, R_{-\alpha}] \subset \mathfrak{t}_{\mathbf{C}}$.

Example $G = SU(n)$.
A Cartan subalgebra is the vector space of diagonal matrices

$$\sqrt{-1}\,\mathrm{diag}(\lambda_1, \dots, \lambda_n),$$

with $\sum \lambda_i = 0$.
It is the Lie algebra of $SL(n, \mathbf{C})$.
The co-ordinates λ_i are linear functions on $\mathfrak{t}$, hence elements of $\mathfrak{t}^*$. The roots are $\lambda_i - \lambda_j$ for $i \neq j$.

The corresponding root space R_α is the one-dimensional space of complex matrices with only component in the (ij) position.

Suppose we fix an element $\xi_0 \in \mathfrak{t}$ such that $\alpha(\xi) \neq 0$ for any root α. Then we say a root is positive if $\alpha(\xi_0) > 0$ and just one of $\alpha, -\alpha$ is positive. Then we have a description of $\mathfrak{g}$ *as a vector space*:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha > 0} R_\alpha$$

Example $G = SU(n)$
Let ξ_0 be the trace-free part of $\mathrm{diag}(1, 2, \dots, n)$. Then if $\alpha = \lambda_i - \lambda_j$ we have $\alpha(\xi_0) = i - j$. So $\alpha > 0$ if $i > j$. The sum of the root spaces for positive roots is the space of strictly upper triangular matrices. A matrix M in $\mathfrak{su}(n)$ is determined by its diagonal part and its upper triangular part—the lower triangular part is fixed by the fact that $M = -M^*$.

In the case of $G = SU(3)$ we can make a picture of the 6 roots as a hexagonal configuration in the plane.

In Section 5 we will discuss the structure of roots etc. further.

Section 4. Representation Theory: The Borel-Weil Theorem

The main point of this section is to discuss a geometric description of the unitary representations of a compact Lie group. This involves an interplay between *complex geometry* and *symplectic geometry*. The result is related to the general theme of *quantisation*.

The mathematical setting for classical mechanics is a symplectic manifold (M, ω).

The mathematical setting for quantum mechanics is a Hilbert space.

In a problem with a symmetry group G we might expect some relation between symplectic manifolds with G-action and unitary representations of G.

Let G be any Lie group. It acts on $\mathfrak{g}^*$, the dual of the Lie algebra, by the co-adjoint action. Let θ be a point in $\mathfrak{g}^*$ and $\Gamma \subset G$ be the stabiliser of θ.
The restriction of θ to $\mathrm{Lie}(\Gamma) \subset \mathfrak{g}$ is a Lie algebra homomorphism $\mathrm{Lie}(\Gamma) \to \mathbf{R}$ (Exercise).
We will say that θ is an *integral point* if this defines a Lie group homomorphism $\sigma : \Gamma \to S^1$.

The Borel-Weil Theorem: Version 1.

If G is compact and connected there is a 1-1 correspondence between irreducible unitary representations of G and integral co-adjoint orbits

If θ is an integral point with orbit $M \subset \mathfrak{g}^*$ we get a complex line bundle L over M, associated to the principle Γ bundle $G \to M$. We will see that M can be given a complex structure and L is a holomorphic line bundle. The representation associated to the orbit will be the space of holomorphic sections of L.

Example $G = SU(2)$. We identify $\mathfrak{g}^*$ with $\mathbf{R}^3$. The non-zero orbits are spheres. With suitable normalisation, the integral orbits are the spheres with integer radii. The line bundle over the sphere of radius $k > 0$ is $\mathcal{O}(k)$ and the holomorphic sections give the symmetric power $s^k(\mathbf{C}^2)$.

The co-adjoint orbits have G-invariant symplectic structures. Let $M \subset \mathfrak{g}^*$ be the orbit of θ. We have a map

$$\mathfrak{g} \to TM_\theta.$$

Given $v_1, v_2 \in TM_\theta$ choose lifts $\xi_1, \xi_2 \in \mathfrak{g}$ and define

$$\omega(v_1, v_2) = \theta([\xi_1, \xi_2]).$$

Exercise This is independent of the choice of lifts and defines a G-invariant symplectic form on M.

Let (N, Ω) be a symplectic manifold with G-action. The Hamiltionian construction gives a map from $C^\infty(N)$ to the Lie algebra of symplectic vector fields. We say the action is Hamiltonian if there is a lift of the Lie algebra action to a homomorphism

$$\mu^* : \mathfrak{g} \to C^\infty(N)$$

(with the Poisson bracket on $C^\infty(N)$). Equivalently, this is a G-equivariant *moment map*

$$\mu : N \to \mathfrak{g}^*.$$

For the co-adjoint orbits this is just the inclusion map.
If the Hamiltonian action on N is transitive then the image of μ is a co-adjoint orbit and one sees that μ is a covering map. So up to possible coverings (which will not occur in our situation) the co-adjoint orbits are exactly the symplectic manifolds with transitive Hamiltonian G-action.

Before continuing our discussion of the Borel-Weil Theorem we digress to recall the algebraic analyis of unitary representations of $SU(2)$.
Fix the standard circle subgroup $S^1 \subset S^2$ with generator iH where

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

So H, X, Y is a basis for the complexified Lie algebra $\mathfrak{sl}(2, \mathbf{C})$ and

$$[H, X] = 2X, [H, Y] = -2Y, [X, Y] = 2H. \qquad (****)$$

Let V be a representation of $SU(2)$. This decomposes under the action of S^1 into a sum of weight spaces $V = \bigoplus V_\theta$ for $\theta \in \mathbf{Z}$. Let k be the "highest weight" (the largest weight with $V_\theta \neq 0$).
By definition we have $Hv = \theta v$ for $v \in V_\theta$. The brackets (****) imply that

$$HXv = (\theta + 2)Xv \quad HYv = (\theta - 2)Yv$$

So

$$X : V_\theta \to V_{\theta+2} \quad , Y : V_\theta \to V_{\theta-2}.$$

From this one sees easily that if V is irreducible the space V_k is one dimensional and if $\mathbf{v}$ is a non-zero element then a basis of V is given by $\mathbf{v}, Y\mathbf{v}, Y^2\mathbf{v}, \ldots, Y^k\mathbf{v}$. The relations (****) completely define the representation in this basis and $V = S^k(\mathbf{C}^2)$.

We return to the Borel-Weil Theorem, so G will be a compact connected Lie group. We want to define two constructions:
Irreducible representation $\rightarrow$ co-adjoint orbit.
co-adjoint orbit $\rightarrow$ irreducible representation.

Preliminary fact. A co-adjoint orbit of G is simply connected. Any class in $\pi_1(G)$ is represented by a geodesic loop at the identity; i.e. a 1-parameter subgroup $S^1 \subset G$. Thus if T is a maximal torus the map $\pi_1(T) \rightarrow \pi_1(G)$ is surjective. For a co-adjoint orbit $M = G/\Gamma$ we have $T \subset \Gamma$ so $\pi_1(\Gamma) \rightarrow \pi_1(G)$ is surjective and $\pi_1(M)$ is trivial.

Preliminary Observation

Suppose (X, ω) is a compact Kahler manifold and v is a vector field on X defining a 1-parameter family $f_t : X \to X$ of holomorphic isometries. Then:

- The family f_t extends to $t \in \mathbf{C}$,
- If H is the Hamiltonian for v then $Iv = \mathrm{grad}\, H$ and f_{it} is the gradient flow of H.
- The critical points of H are the fixed points of f_t and the Hessian of H at a critical point p generates the action of f_t on TX_p. Thus the eigenspaces of the Hessian are complex subspaces of TX_p.
- $|v|^2 = \nabla_{Iv} H$.

Fix an invariant Euclidean norm on $\mathfrak{g}$ and identify $\mathfrak{g}$ with its dual.
Lemma *Let X be a compact Kahler manifold with a Hamilonian G-action which preserves the complex structure. Let $p \in X$ be a point where $\mu|^2$ is maximal. Then the G-orbit of p is a complex submanifold of X.*
The complexified group G^c acts holomorphically on X. It suffices to show that the G^c orbit of p is the same as the G-orbit. Let $\Gamma \subset G$ be the stabiliser of p and $\xi = \mu(p) \in \mathfrak{g}$.

Let U be the orthogonal complement of $\mathrm{Lie}(\Gamma)$ in $\mathfrak{g}$. The derivative of the action gives a linear map from U to TX_p. This defines a complex linear map $\rho : U \otimes \mathbf{C} \to TX_p$. The map ad_ξ induces a complex linear map from $U \otimes \mathbf{C}$ to itself. Write

$$U \otimes \mathbf{C} = U_0 \oplus U_+ \oplus U_-$$

where ad_ξ is zero on U_0 and $U_\pm$ is the span of the $\pm$ eigenspaces of $i\mathrm{ad}_\xi$. The spaces $U_\pm$ are complex conjugate.

We claim that

- ρ vanishes on U_-;
- $U_0 = 0$.

Let v' be the vector field on X defined by $i\xi$ under the complexified action. It is the gradient vector field of the Hamiltonian $H = \langle \mu, \xi \rangle$.
Let $\eta \in U_-$ be an eigenvector for $i\mathrm{ad}_\xi$, so $i[\xi, \eta] = \lambda\eta$ for $\lambda < 0$. Under the complexified action, η defines a holomorphic vector field N on X and $[v', N] = \lambda N$.
For the first item, we want to show that N vanishes at p.
We have the formula $[v', N] = \nabla_{v'} N - \nabla_N v'$. Since v' vanishes at p we have $\nabla_N v' = -\lambda N$ at the point p.
Since $v' = \mathrm{grad}\, H$, the quantity $\nabla_N v'$ is the Hessian of H at the critical point p, evaluated in the direction N.
The point p is a maximum point for H, so the Hessian is negative and this implies that $N(p) = 0$.

For the second item, suppose $\eta \in \mathfrak{g}$ is a real element of W_0, so $[\xi, \eta] = 0$. Let Z be the vector field on X defined by $i\eta$. We want to show that Z vanishes at p. (Because then $\eta \in \mathrm{Lie}(\Gamma)$ so $\eta = 0$.)

We have $[v', Z] = 0$ so, as in the first item, $\nabla_Z v' = 0$ at the point p and $Z(p)$ lies in the null-space of the Hessian of H.

Let $F = \langle \mu, \eta \rangle$ be the Hamiltonian generating $-IZ$. Then $\nabla_Z F = |Z|^2$.

Suppose that $Z(p) \neq 0$. Then $\nabla_Z F$ does not vanish at p. Since Z lies in the null-space of the Hessian of H, this implies that the second derivative of $|\mu|^2$ in the direction Z is positive, which contradicts the fact that $|\mu|^2$ is maximised at p.

Let V be a unitary representation of G and consider the action on $X = \mathbf{P}(V^*)$ with its standard symplectic structure ω_{FS} This action is Hamiltonian. It suffices to consider the action of $U(V)$ on $\mathbf{P}(V)$. The moment map is given by

$$\mu(z) = \frac{i}{|z|^2} z z^*.$$

By the Lemma, there is an orbit $M \subset \mathbf{P}(V^*)$ which is a complex submanifold. Then ω_{FS} defines a symplectic structure on M and the moment map is the restriction of $\mu_{\mathbf{P}}$. Since the action is transitive the moment map identifies M with a co-adjoint orbit. It is an integral orbit with the action on the fibre of $\mathcal{O}(1)$ over p defining a homomorphism $\Gamma \to S^1$.

Now we want to see how to go from an integral co-adjoint orbit to a representation.
Choose a maximal torus $T \subset G$. Any co-adjoint orbit M can be represented by a point ξ in $\mathfrak{t} = \mathfrak{t}^*$. For exposition, suppose first that ξ is "generic", not orthogonal to any root and so defines a set of positive roots. Write

$$\mathfrak{g} \otimes \mathbf{C} = \mathfrak{t} \otimes \mathbf{C} \oplus W_+ \oplus W_-$$

where $W_\pm$ are spanned by the $\pm$ root spaces.

In this case the stabiliser Γ is the torus T. The tangent space of M at the base point ξ is identified with the real part of $W^+ \oplus W^-$ which we can identify with W^+. So we get an almost-complex structure on M. The subspace $\mathfrak{t} \otimes \mathbf{C} \oplus W_+$ is a Lie subalgebra of $\mathfrak{g} \otimes \mathbf{C}$. It defines a complex subgroup $B \subset G^c$. Since $G \subset G^c$ and $T \subset B$ we have a map $G/T \to G^c/B$ and looking at the Lie algebras we see that this is an equivalence. So the co-adjoint orbit M has a complex description as G^c/B and this makes it clear that the almost-complex structure is integrable.
Similarly we have a homomorphism $B \to \mathbf{C}^*$ from which one see that the line bundle $L \to M$ is a holomorphic line bundle. Thus we get a representation of G on the vector space $H^0(M, L)$.

For $G = SU(n)$ we get the flag manifolds discussed before. The generic case above is when all eigenvalues are distinct and G/T is the space of maximal flags

$$0 \subset E_1 \subset \ldots E_r \cdots \subset E_{n-1} \subset \mathbf{C}^n$$

with $\mathrm{dim} E_i = i$. The subgroup B is the group of upper triangular matrices, preserving the standard flag.
The co-adjoint orbits of $SU(n)$ are flag manifolds and have a complex description $SL(n, \mathbf{C})/P$ where P is a group of block upper triangular matrices.
The same picture applies for general G. The coadjoint orbit G/Γ has a complex description G^c/P where the Lie algebra of P includes the elements of W_- corresponding to roots α with $\alpha(\xi) = 0$.

Borel-Weil Theorem; Version 2
These two constructions are inverse (when the first construction is restricted to irreducible representations V).

The proof involves checking various things.

The main point is to show that the space $H^0(M, L)$ is non-trivial. In fact contains a particular "highest weight vector".

Suppose (X, ω) is a symplectic manifold and $L \to X$ is a Hermitian line bundle with connection and curvature $-i\omega$.

We claim that the Lie algebra $C^\infty(X)$ acts on the line bundle. Let f be a function on X defining a vector field v. Define an operator on sections of L:

$$D_f(s) = \nabla_v s - ifs.$$

Then if g is another function you find that $[D_f, D_g] = \{f, g\}$. Thus *a moment map for the action of a group G on (M, ω) defines a lift of the action to L.*

Consider the case $G = S^1$ with Hamiltonian H. The fixed points of the action are the critical points of H. The value $H(p)$ at a critical point p is the weight of the action on the fibre L_p.

Now suppose that (X, ω) is a compact Kahler manifold, the line bundle $L \to X$ is a holomorphic line bundle and S^1 acts non-trivially on X and L. The space of sections $H^0(L)$ is a representation of S^1 and so decomposes into a sum of weight spaces.

Proposition *If H is a Morse function and the image of $H : X \to \mathbf{R}$ is the interval $[a, b]$ then:*

- *The weights lie in $[a, b]$.*
- *a and b are weights and the corresponding weight space are 1-dimensional.*

We can apply this Proposition to 1-parameter subgroups in G to show that our representations $H^0(M, L)$ are non-trivial.

Without loss of generality, $b = 0$. The indices of all critical points are even, so by elementary Morse Theory there is a unique maximum $p \in X$. The circle acts trivially on the fibre L_p. Choose a basis element $e \in L_p$. We want to find an S^1-invariant holomorphic section s with $s(p) = e$.
Let $f_t : X \to X$ be the increasing gradient flow of H. This is the same as the complexified action restricted to $\mathbf{R}^+ \subset \mathbf{C}^*$. For any point $x \in X$ the flow $f_t(x)$ converges to a critical point of H as $t \to \infty$. For generic x this limit is p. In fact there is a complex analytic subvariety $X \subset Z$ such that this is true for $x \in X \setminus Z$. Consider the lifted action to the total space of L. This gives $\tilde{f}_t$ covering f_t. For $x \in X \setminus Z$ we define $s(x) \in L_x$ to be the unique point such that $\lim_{t\to\infty} \tilde{f}_t(s(x)) = e$.
This defines a holomorphic section s of L over $X \setminus Z$.

To see that s extends holomorphically over Z let $\mathbf{P}$ be the standard compactification of the total space of L:

$$\mathbf{P} = \mathbf{P}(L \oplus \mathbf{C}).$$

We have a zero section $X_0 \subset \mathbf{P}$ and an infinity section $X_\infty \subset \mathbf{P}$. The section s defines a complex submanifold Σ of $\mathbf{P}$ which is not closed. We claim that the closure $\overline{\Sigma}$ does not intersect X_∞. This implies that $|s|$ is bounded on $X \setminus Z$ which means that s extends holomorphically over Z by a standard extension theorem in complex analysis.
In fact the $\overline{\Sigma} \setminus \Sigma$ lies in the X_0 so the extended holomorphic section vanishes on Z.

By general Morse Theory, any point σ in $\overline{\Sigma}$ can be joined to e by a "chain" of flow lines of $\tilde{f}_t$. For example $\lim_{t\to\infty}\tilde{f}_t(\sigma) = q$ where q is a fixed point of the action on **P** and there is a doubly infinite flow line $f_t(\sigma')$ with

$$\lim_{t\to-\infty}\tilde{f}_t(\sigma') = q \qquad \lim_{t\to\infty}\tilde{f}_t(\sigma') = e.$$

Apart from the fibre over p, the fixed points on **P** are the points in X_0, X_∞ lying over the other critical points p' of H. Since $H(p') < 0$ the point at infinity is an attractive point for flow $\tilde{f}_t$ as $t \to \infty$ on the fibre over p'. Simple considerations then show that $\overline{\Sigma} \setminus \Sigma$ lies in X_0.

Suppose s' is another invariant section. Then s'/s is an invariant holomorphic function in a neighbourhood of p. Looking at the Taylor series you see that this is a constant, so the weight space is 1-dimensional.

Similarly, looking at the Taylor series around p, you see that there is no section with weight strictly greater than 0.

Example

For $G = SU(3) = SU(V)$ the generic co-adjoint orbit is the flag manifold

$$\mathbf{F} = \{(x, \xi) \in \mathbf{P} \times \mathbf{P}^* : \xi.x = 0\},$$

where $\mathbf{P} = \mathbf{P}(V)$ is the projective plane and $\mathbf{P}^*$ is the dual plane. Write $\mathcal{O}(p, q)$ for the line bundle

$$\pi_1^*(\mathcal{O}(p)) \otimes \pi_2^*(\mathcal{O}(q))$$

over $\mathbf{F}$. The irreducible representations $V_{p,q}$ of $SU(3)$ are the sections of $\mathcal{O}(p, q)$ for $p, q \geq 0$. When one of p, q vanishes they can also be described as sections over $\mathbf{P}$ or $\mathbf{P}^*$, which are the smaller co-adjoint orbits. Explicitly, $V_{p,q}$ is the quotient of $s^{p,q} = s^p(V^*) \otimes s^q(V)$ by the image of the natural map $s^{p-1,q-1} \to s^{p,q}$. Alternatively, this can be identified with the kernel of a natural map $s^{p,q} \to s^{p-1,q-1}$.

Section 5: More on structure theory and representations

Proposition *The only compact connected Lie groups of rank* 1 *are* $S^1, SU(2), SO(3)$.

To see this we easily reduce to the case when the Lie algebra has trivial centre. Then the non-zero coadjoint orbits have codimension 1 and so are spheres in the Lie algebra. The only sphere which carries a symplectic structure is S^2 so the group has dimension 3 and then the argument is straightforward.

Go back to our compact connected Lie group G with maximal torus T and decomposition

$$\mathfrak{g} \otimes \mathbf{C} = \mathfrak{t} \otimes \mathbf{C} \oplus \bigoplus R_\alpha$$

Proposition

If α is a root then:

- the only multiple $k\alpha$ which is a root, for non-zero k, is $k = \pm 1$.
- the root space R_α is one dimensional and $\mathbf{R}\alpha + \Re(R_\alpha)$ is a Lie algebra isomorphic to $\mathfrak{su}(2)$. (equivalently $\mathbf{C}\alpha + R_\alpha + R_{-\alpha}$ is a copy of $\mathfrak{sl}(2, \mathbf{C})$.)

Suppose $r_\alpha \in R_\alpha$ and $r_{-\alpha} \in R_{-\alpha}$. If $\eta \in \mathfrak{t}$ is orthogonal to α then

$$\langle \eta, [r_\alpha, r_{-\alpha}] \rangle = -\langle [r_\alpha, \eta], r_{-\alpha} \rangle = 0$$

since

$$[\eta, r_\alpha] = \langle \eta, \alpha \rangle r_\alpha$$

by the definition of the root space. Thus $[r_\alpha r_{-\alpha}]$ is a multiple of α. This implies that

$$\mathbf{C}\alpha \oplus \bigoplus_k R_{k\alpha}$$

is a Lie algebra of rank 1. Then the Proposition follows from the previous result.

Recall that the *normaliser* $N(T)$ of T is the set of $g \in G$ such that $gTg^{-1} = T$. It is a subgroup and $T \subset N(T)$ is a normal subgroup.
The *Weyl group* W is the quotient $N(T)/T$. It is a finite group and it acts on T and its Lie algebra $\mathfrak{t}$. It also acts on the finite set of roots in $\mathfrak{t}$.

Example $G = SU(n)$ or $U(n)$. With our standard maximal torus of diagonal matrices, the Weyl group is the permutation group of order $n!$. It acts on the roots $\lambda_i - \lambda_j$.
In particular, for $G = SU(2)$ the Weyl group has order 2 and acts on $\mathfrak{t} = \mathbf{R}$ as multiplication by ± 1.

Let V be an irreducible unitary representation of G. Restriction gives a representation of T which decomposes

$$V = \bigoplus V_\mu$$

where the weights μ lie in the weight lattice $\Lambda^* \subset \mathfrak{t}$. The multiplicity of a weight μ is $\dim V_\mu$
(Recall that we are identifying $\mathfrak{t}$ with its dual.)

From the definitions, the weights and multiplicities are preserved by the action of W on $\mathfrak{t}$.

Each weight has a norm. Let μ_0 be a weight of maximal norm.

Proposition

The weights of V lie in the intersection of Λ^* with the convex hull of the W-orbit of μ_0.

Algebraic approach

For each root α fix a basis element $r_\alpha \in R_\alpha$. In the representation on V this maps to an element of $\mathrm{End}\ V$. For $\eta \in \mathfrak{t}$ and $v_\mu \in V_\mu$ we have

$$\eta(r_\alpha v_\mu) = [\eta, r_\alpha] v_\mu + r_\alpha \eta(v_\mu).$$

Using the definitions of the root and weight spaces this is

$$\langle \eta, \alpha \rangle r_\alpha(v_\mu) + \langle \eta, \mu \rangle r_\alpha v_\mu = \langle \eta, \alpha + \mu \rangle r_\alpha v_\mu.$$

So r_α is either 0 on V_μ or maps V_μ to another weight space $V_{\mu+\alpha}$.

Assume for simplicity that μ_0 is not orthogonal to any root. So it defines a set of positive roots. For each positive root α let $X_\alpha : V \to V$ be the action of r_α and $Y_\alpha : V \to V$ be that of $r_{-\alpha}$. By the norm-maximising condition, for any positive root α the sum $\alpha + \mu_0$ is *not* a weight. So X_α vanishes on V_{μ_0}. Fix a basis element $\mathbf{v} \in V_{\mu_0}$.

As in the case of $\mathfrak{sl}(2, \mathbf{C})$, the action of the products of the Y_α on $\mathbf{v}$ generates a $\mathfrak{g}$- invariant subspace of V which must be all of V.

Let $C^* \subset \mathfrak{t}$ be the convex cone generated by the positive roots:

$$C^* = \{\sum t_\alpha r_\alpha : t_\alpha \geq 0\}.$$

It follows from the discussion above that the weights all lie in

$$v_{\mu_0} - C^*.$$

But the set of weights is preserved by the Weyl group, so lies in

$$\bigcap_{w \in W} w(v_{\mu_0}) - C^*). \qquad (****)$$

In general, let K be the convex hull of a finite set P of points in $\mathbf{R}^n$. For each $p \in P$ there is a *tangent cone* TK_p of K at p. This is a convex cone such that a neighbourhood of p in K is the same as a neighbourhood in $p - TK_p$. One can show that

$$K = \bigcap_{p \in P} p - TK_p.$$

In the case at hand, the discussion below of reflections and simple roots shows that

$$C^* \subset TK_{\mu_0}.$$

Then (****) implies that the set of weights is contained in the convex hull K.
(In fact the set of weights is exactly the intersection of the weight lattice with the convex hull.)

Symplectic geometry approach

Suppose an m-dimensional torus T acts on a compact symplectic manifold X with moment map $\mu : X \to \mathbf{R}^m$. Let $F \subset X$ be the set of points fixed by T. By the definition of the moment map, it is constant on each connected component of F so $\mu(F)$ is a finite set in $\mathbf{R}^m$.

Theorem (Atiyah, Guillemin, Sternberg) *The image $\mu(X)$ is the convex hull of $\mu(F)$.*

Now suppose that X is Kahler and there is a holomorphic line bundle $L \to X$ as we considered before. So the vector space $H^0(X, L)$ is a representation of T and decomposes as a sum of weight spaces H_μ for $\mu \in \mathbf{Z}^m \subset \mathbf{R}^m$.

Theorem *The weights lie in the convex set $\mu(T)$ and each point in $\mu(F)$ is a weight.*

These results can be proved without too much difficulty by reducing to the case $m = 1$ we studied before.

We can apply these results to a co-adjoint orbit $M \subset \mathfrak{g}$ and the restriction of the G action to $T \subset G$. The moment map $\mu : M \to \mathfrak{t}$ is the restriction of orthogonal projection $\mathfrak{t} \subset \mathfrak{g}$. We have

Proposition

- The fixed points $F \subset M$ are the intersection $M \cap \mathfrak{t}$;
- The fixed points form one orbit of the action of W on $\mathfrak{t}$.

Given this and the two Theorems above, the statement about weights of an irreducible representation of G follows from the Borel-Weil Theorem.

Example

$G = SU(n)$ and M is the orbit of $\mathrm{diag}(\lambda_i)$ for λ_i distinct. This can be identified with direct sum decompositions into 1-dimensional subspaces

$$\mathbf{C}^n = P_1 \oplus P_2 \oplus P_n.$$

The standard torus fixes such a decomposition if and only if the P_i are a permutation of the standard co-ordinate axes in $\mathbf{C}^n$.

The orthogonal complement of any root α is a "root plane" the hyperplane $\alpha^\perp \subset \mathfrak{t}$. A *Weyl chamber* is the closure in $\mathfrak{t}$ of a connected components of the complement of all these hyperplanes. Any Weyl chamber is defined by a set of inequalities

$$\{\eta \in \mathfrak{t} : \langle \eta, \alpha \rangle \geq 0, \alpha \in S\}$$

for some subset S of roots. Thus the Weyl chambers are convex cones.

Proposition *For each root α the reflection in the hyperplane $\alpha^\perp$ is in the Weyl group.*

The proof goes by reducing to the case of $SU(2)$ using the copy of $\mathfrak{su}(2)$ associated to each root.

Example Take $G = SU(n)$ and the root $\lambda_1 - \lambda_2$. In the permutation group on n elements the reflection corresponds to a transposition (12). This is realised in the copy of $SU(2) \subset SU(n)$ acting on the first two co-ordinates.

Proposition

The Weyl group acts simply transitively on the set of Weyl chambers. Each orbit in $\mathfrak{t}$ *of the Weyl group meets each Weyl chamber exactly once*

Fix one Weyl chamber C and call it the *fundamental Weyl chamber*(FWC) . So we have equivalences between:

- integral co-adjoint orbits in $\mathfrak{g}$;
- orbits of the Weyl group acting on the lattice of weights in $\mathfrak{t}$;
- lattice points in the fundamental Weyl chamber.

The main fact in the representation theory is that any one of these sets is equivalent to the set of isomorphism classes of irreducible representations.

The lattice point corresponding to a co-adjoint orbit is the "highest weight". It is characterised by any of

- It maximises the norm over all weights of the representation in the FWC;
- It is maximal in the partial order induced by inclusion of convex hulls of W-orbits;
- Fix any $\eta \in \mathfrak{t}$ with $L_\eta(\xi) = \langle \eta, \xi \rangle \geq 0$ for all $\xi \in C$. Then the highest weight maximises the linear functional L over all the weights.

The choice of fundamental Weyl chamber gives a choice of a set of positive roots. A root α is positive if $L_\alpha \geq 0$ on C.

To sum up:

Theorem *For each lattice point $\mu_0 \in C \subset \mathfrak{t}$ there is a unique irreducible representation with highest weight μ_0 and these are all the irreducible representations of G.*

For each point η in the interior of the Weyl chamber C the co-adjoint orbit can be identified with G/T, with the same complex structure. Suppose that $\mathfrak{g}$ has trivial centre. Then $H^2(G/T; \mathbf{R}) = \mathfrak{t}$ and $H^2(G/T; \mathbf{Z})$ can be identified with the weight lattice in $\mathfrak{t}$. For each point λ in the weight lattice we get a holomorphic line bundle L_λ over G/T. This is a positive line bundle when λ lies in the interior of C.

The Borel-Weil Theorem: Version 3 *The holomorphic sections of $L_\mu \to G/T$ define the irreducible representation of G with highest weight $\mu \in C$.*

The simple roots are the roots defining the walls of the fundamental Weyl chamber.

- Any positive root is a sum of simple roots with positive coefficients;
- If $\mathfrak{g}$ has trivial centre, the number of simple roots is the rank of G, i.e. the dimension of $\mathfrak{t}$. The Weyl chamber is affine-equivalent to a cone over a simplex.

For $SU(n)$ or $U(n)$ we can take as FWC the set defined by

$$\lambda_1 \leq \lambda_2 \leq \lambda_n$$

The positive roots are $\lambda_i - \lambda_j$ for $i < j$.There are $(n-1)$ simple roots

$$\lambda_2 - \lambda_1, \lambda_3 - \lambda_2, \lambda_n - \lambda_{n-2}.$$

For $SO(2n)$ we take the maximal torus of block-diagonal matrices with blocks

$$\begin{pmatrix} \cos\lambda_i & -\sin\lambda_i \\ \sin\lambda_i & \cos\lambda_i \end{pmatrix}.$$

So we have co-ordinates $\lambda_1, \dots \lambda_n$. The Weyl group is generated by permutations of the λ_i and change of sign of an *even*number of the λ_i. A FWC is the set defined by

$$-\lambda_2 \leq \lambda_1 \leq \lambda_2 \dots \leq \lambda_n.$$

The roots are $\pm\lambda_i \pm \lambda_j$ for $i \neq j$. The simple roots are

$$\lambda_1 + \lambda_2, \lambda_2 - \lambda_1, \lambda_3 - \lambda_2, \dots \lambda_n - \lambda_{n-1}$$

For $SO(2n+1)$ we can take the same maximal torus, adding an entry 1 on the diagonal. The Weyl group is generated by permutations and all changes of sign. The roots are $\pm\lambda_i \pm \lambda_j$ and $\pm\lambda_i$. A fundamental chamber is defined by

$$0 \leq \lambda_1 \leq \lambda_2 \cdots \leq \lambda_n.$$

The simple roots are

$$\lambda_1, \lambda_2 - \lambda_1, \lambda_3 - \lambda_2, \ldots.$$

Co-adjoint orbits for the orthogonal groups

Let ξ be the matrix in the maximal torus of $SO(m)$ with one non-trivial block. The co-adjoint orbit of ξ is the Grassmannian $Gr_2(\mathbf{R}^m)$ or oriented 2-planes in $\mathbf{R}^m$. Let v_1, v_2 be an oriented orthonormal basis for a 2-plane and set $Z = v_1 + iv_2 \in \mathbf{C}^m$. Then $\sum Z_i^2 = 0$ and Z defines a point in the standard quadric $Q \subset \mathbf{CP}^{m-1}$. This exhibits the co-adjoint orbit as a complex manifold.

In general, a co-adjoint orbit of $SO(m)$ can be identified with a set of "symmetric flags"

$$E_0 \subset E_1 \subset E_2 \subset E_r \subset \mathbf{C}^m$$

where E_i is the annihilliator of E_{r-i} with respect to the standard quadratic form on $\mathbf{C}^m$. The case above is when $r = 1$ and $\dim E_0 = 1$. Then the condition is just that the form vanishes on E_0.

Consider the case of $SO(5)$. We have simple roots $\lambda_1, \lambda_1 - \lambda_2$ as above. The coadjoint orbit corresponding to an interior point of the chamber is the set of pairs (L, p) where L is a line in the quadric $Q \subset \mathbf{CP}^4$ and p is a point in L. On one edge of the chamber the orbit becomes just Q and on the other the lines in Q.

Constraints on roots

Let $\Pi \subset \mathfrak{t}$ be a plane containing at least two independent roots. Then $\Pi \otimes \mathbf{C} \oplus \bigoplus_{\alpha \in \Pi} R_\alpha$ is a Lie subalgebra of rank 2. There are just four possibilities.

Case 0 The roots in Π are $\pm\alpha, \pm\beta$ with α, β orthogonal.

Case 1 There are 6 roots in Π, they all have the same length and going around the unit circle make successive angles $\pi/3$.

Case 2 There are 8 roots in Π, going around the unit circle they make successive angles $\pi/4$ and the lengths are alternately $L, \sqrt{2}L$ for some L.

Case 3 There are 12 roots in Π, going around the unit circle they make successive angles $\pi/6$ and the lengths are alternately $L, \sqrt{3}L$ for some L.

The proof uses the fact that the reflections in the roots preserves the lattice spanned by the roots.
This classification is closely related to the discussion of elliptic curves with “complex multiplication”.

- Case 0 occurs for $G = SO(4)$ or $G = SU(2) \times SU(2)$.
- Case 1 occurs for $G = SU(3)$.
- Case 2 occurs for $G = SO(5)$.
- Case 3 occurs for the exceptional Lie group G_2 that we discuss in the next Section.

The *Dynkin diagram* is formed by taking one node for each simple root and joining nodes by k bonds according to cases $k = 0, 1, 2, 3$. We also record which roots are "long" and "short"

Example

The simple roots for $SU(n)$ can be taken as

$$\lambda_2 - \lambda_1, \lambda_3 - \lambda_2, \ldots, \lambda_n - \lambda_{n-1}.$$

They all have length $\sqrt{2}$. If $n > 3$ the first and last are orthogonal and the angle between successive roots in the list is $2\pi/3$. Changing one sign we get roots with angle $\pi/3$. The Dynkin diagram is a chain of $n - 1$ nodes joined by 1 bond. This also holds for $n = 2, 3$.

Example

The simple roots for $SO(2n+1)$ can be taken as

$$\lambda_1, \lambda_1 - \lambda_2 \ldots, \lambda_n - \lambda_{n-1}.$$

The first is shorter than the others. The diagram is a chain of n nodes, the first two joined by 2 bonds and the rest by 1.

The diagram for $Sp(n)$ is the same as for $SO(2n+1)$, except that the first root is long and the rest are short.

The Weyl character formula

Let $\rho : G \to U(V)$ be a representation of our compact Lie group G. The character is the function on G:

$$\chi_V(g) = \mathrm{Tr}\rho(g).$$

It is a "class function", preserved by conjugation in G.
Fix a maximal torus. Since any element of G is conjugate to one in T the character is determined by its restriction to T. This is invariant under the action of the Weyl group W.
We have a decomposition into weight spaces $V = \bigoplus_\lambda V_\lambda$ with multiplicities $m_\lambda = \dim V_\lambda$. Think of a function on T as a periodic function on $\mathfrak{t}$ and write e_λ for the function

$$e_\lambda(\theta) = e^{i\langle\theta,\lambda\rangle}.$$

Then the restricted character is $\sum_\lambda m_\lambda e_\lambda$.
(It is the Fourier transform of the collection of multiplicities, regarded as a measure on $\mathfrak{t}$).

Example $G = SU(2)$, ρ the irreducible representation of dimension $k+1$. The weights are $-k, -k+2, \dots, k-2, k$ all with multiplicity 1 and the character is

$$\chi(\theta) = e^{-ik\theta} + e^{-i(k-2)\theta} + \cdots + e^{ik\theta}$$

Fix a positive Weyl chamber C, so the irreducible representations are parametrised by lattice points in C. Write

$$D(\theta) = \prod_{\alpha>0} e_\alpha(\theta/2) - e_{-\alpha}(\theta/2) = \prod_{\alpha>0} e^{i\alpha.\theta/2} - e^{-i\alpha.\theta/2}$$

which is

$$\prod_{\alpha>0} 2i\sin(\alpha.\theta/2).$$

(Here the product is over the positive roots.)

This is a periodic function on $\mathfrak{t}$; it is not invariant under the Weyl group W but *alternating*. Applying reflection in a root changes the sign of D.

For $w \in W$ write $(-1)^w \in \{\pm 1\}$ for the determinant of the action of w on $\mathfrak{t}$. For any weight λ define the alternating sum:

$$A_\lambda = \sum_{w \in W} (-1)^w e_{w(\lambda)}.$$

Then one shows that $D = A_\rho$ where $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$.

The Weyl character formula

For the representation V with highest weight $\lambda \in C$ the restriction of the character to T is

$$\chi_\lambda = \frac{A_{\lambda+\rho}}{A_\rho}.$$

For example in the case of $SU(2)$

$$\frac{e^{i(k+1)\theta} - e^{-i(k+1)\theta}}{e^{i\theta} - e^{-i\theta}} = e^{ik\theta} + e^{i(k-2)\theta} + \cdots + e^{-ik\theta}.$$

We outline two proofs of the character formula.

Weyl's proof

This uses general facts about representations of compact groups.

Let G be a compact group with bi-invariant measure, normalised to $\mathrm{Vol}(G) = 1$. G acts by left and right translation on $L^2(G)$.

Let V be a finite dimensional unitary representation of G. Then G acts on the left and right on $\mathrm{End}\ V = V \otimes V^*$. The basic fact is that

$$L^2(G) = \bigoplus_V \mathrm{End}\ V,$$

where the sum runs over isomorphism classes of irreducible representations.

This implies that the characters χ_V form an orthonormal basis for the class functions.

Weyl's integration formula

For a class function f on G;

$$\int_G |f|^2 = |W|^{-1} \int_T |f|^2 |D|^2.$$

This can be seen as a special case of a formula in Riemannian geometry for the volume element in exponential coordinates.

Now for a weight $\lambda \in C$ write $\tilde{\chi}_\lambda = \frac{A_{\lambda+\rho}}{A_\rho}$, so we want to prove that $\chi_\lambda = \tilde{\chi}_\lambda$. Since

$$|D|^2 = A_\rho \overline{A_\rho},$$

we have

$$\int_G |\tilde{\chi}_\lambda|^2 = |W|^{-1} \int_T A_\rho \overline{A_\rho} = 1.$$

This uses the usual orthonormal property of the functions e_μ on T and the fact that $\lambda + \rho$ is in the interior of C, so the W orbit contains $|W|$ points.

Similarly $\tilde{\chi}_\lambda, \tilde{\chi}_\mu$ are orthogonal if $\lambda \neq \mu$.

Define a partial order on the weights by $\mu \leq \mu'$ if the *W*-orbit of μ is contained in that of μ'. Then

- the $\tilde{\chi}_\mu$ and χ_μ are both orthonormal systems;
- they are *W*-invariant and have the same highest order term.

It follows by an induction argument that $\chi_\lambda = \tilde{\chi}_\lambda$ for all λ.

Atiyah and Bott proof.

For simplicity consider the case when λ is in the interior of the chamber C, so the co-adjoint orbit is $X = G/T$. Each $\theta \in \mathfrak{t}$ defines a holomorphic diffeomorphism $f_\theta : X \to X$ and for generic θ there are $|W|$ fixed points corresponding to the W-orbit of λ in $\mathfrak{t}$.

As a digression, consider first the topological Lefschetz number $L(f_\theta)$ of f_θ. Since f_θ is homotopic to the identity this is the Euler characteristic of X. The Lefschetz fixed point formula gives

$$L(f_\theta) = \sum_p \mathrm{signdet}\,(1 - df_\theta),$$

where the sum runs over the fixed points p and

$$df_\theta : TX_p \to TX_p.$$

Since df_θ is complex linear the signs are all positive and we conclude that $e(X) = |W|$.

This argument can be promoted to a Morse Theory argument which shows that all the homology of X lies in even dimensions, so the sum of the Betti numbers is $|W|$.

Example If $G = SU(3)$ the flag manifold $SU(3)/T^2$ has

$$b_0 = b_6 = 1 \quad b_2 = b_4 = 2.$$

Atiyah and Bott established a general *holomorphic Lefschetz formula*. Let Z be a compact complex manifold and $E \to Z$ a holomorphic vector bundle. Let $f : Z \to Z$ be a holomorphic diffeomorphism with isolated transverse fixed points. Let $\phi : E \to f^*(E)$ be lift to E. Then f, ϕ act on the sheaf cohomology $H^*(Z, \mathcal{O}(E))$ giving a holomorphic Lefschetz number $L_{\mathcal{O}}(f, \phi)$. The fixed point formula is

$$L_{\mathcal{O}}(f, \phi) = \sum_p \frac{\mathrm{Tr}\ \phi_p}{\det(1 - df_p)}.$$

Apply this to $f_\theta : X \to X$ and the line bundle $L \to X$ defined by the weight λ. It can be shown that the higher cohomology vanishes so the Lefschetz number gives the character of the representation on holomorphic sections of L. But note that a left action of G on X induces a right action on sections of L so we get

$$L(f_{-\theta}) = \chi(\exp(\theta)).$$

Consider the contribution from the fixed point p_0 corresponding to $\lambda \in C$. At this point TX is identified with the direct sum of the positive root spaces and, with $f = f_{-\theta}$,

$$\det(1 - df) = \prod_{\alpha>0}(1 - e^{-i\alpha.\theta}).$$

This is equal to

$$e^{-i\rho.\theta}\prod_{\alpha>0}(e^{i\alpha.\theta/2} - e^{-i\alpha.\theta/2}) = e_{-\rho}(\theta)D(\theta) = e_{-\rho}(\theta)A_\rho(\theta)$$

The trace of ϕ_{p_0} is just $e_\lambda(\theta)$.
So the contribution from p_0 to the fixed point formula is the same as the contribution from $\lambda + \rho$ in the sum $A_{\lambda+\rho}/A_\rho$.
A little thought shows that the same is true for the other fixed points.

Remark

Let $f : \mathbf{C} \to \mathbf{C}$ be the map $f(z) = az$. Then f acts on the polynomial functions on $\mathbf{C}$, $f^*(z^p) = a^p z^p$. So there is a formal trace of the action on all polynomials

$$\mathrm{Tr} f^* = 1 + a + a^2 + \cdots = (1-a)^{-1} = \det(1 - df)^{-1}.$$